Simple Local Search is a PTAS for Feedback Vertex Set in Minor-free Graphs*

Hung Le[1] and Baigong Zheng[1]

1 **School of Electrical Engineering and Computer Science**
Oregon State University, Corvallis, OR, USA
lehu@onid.oregonstate.edu
zhengb@oregonstate.edu

Abstract

We show that simple local search gives a polynomial time approximation scheme (PTAS) for the Feedback Vertex Set (FVS) problem in minor-free graphs. An efficient PTAS in minor-free graphs is known for this problem by Fomin, Lokshtanov, Raman and Saurabh [14]. However, their algorithm uses many advanced tools such as contraction decomposition framework, Courcellei's theorem and the Robertson and Seymour decomposition. Local search, on the contrary, is conceptually simple and easy to implement. It keeps exchanging a constant number of vertices to improve the current solution until a local optimum is reached. We first show that local search yields PTAS for FVS problem in bounded genus graphs, using two classic tools: separator theorem and a bound on maximal matching in bounded genus graphs. We then show a similar result for bounded treewidth graphs, using an amortized argument. Finally, we combine the two arguments to show the PTAS result for minor-free graphs.

1998 ACM Subject Classification G.2.2 Graph Algorithms

Keywords and phrases feedback vertex set, PTAS, local search, minor-free graphs, matching

Digital Object Identifier 10.4230/OASIcs.SOSA.2016.23

1 Introduction

Given an undirected graph, the *Feedback Vertex Set* (FVS) problem asks for a minimum set of vertices such that after removing this set, the resulting graph has no cycle. This problem arises in a variety of applications, including deadlock resolution, circuit testing, artificial intelligence, and analysis of manufacturing processes [13]. Because of its importance, the FVS problem has been studied for a long time in the algorithm area. It is one of Karp's 21 NP-complete problems [20] and is proved to be NP-hard even in planar graphs [32]. The current best approximation ratio for FVS in general graphs is 2 due to Becker and Geiger [3] and Bafna, Berman and Fujito [1].

For some special classes of graphs, better approximation algorithms are known. A polynomial-time approximation scheme (PTAS) is an algorithm that for any fixed $\epsilon > 0$, finds an $(1+\epsilon)$-approximation of the optimal solution in polynomial time. Kleinberg and Kumar [22] gave the first PTAS for FVS problem in planar graphs, followed by a PTAS by Demaine and Hajiaghayi [10] which is generalizable to bounded genus graphs and single-crossing-minor-free graphs. Recently, Cohen-Addad et al. [7] gave a PTAS for the weighted version of this problem in bounded-genus graphs. The most general result obtained by

* This material is based upon work supported by the National Science Foundation under Grant No. CCF-1252833.

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Acces; Article No. 23; pp. 23:1–23:22
Open Access Series in Informatics
OASICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Fomin, Lokshtanov, Raman and Sauraubh [14] was for minor-free graphs. However, all known algorithms for this problem are complicated in both implementation and analysis. We show that a simple local search algorithm (Algorithm 1) gives a PTAS for FVS problem in minor-free graphs.

Algorithm 1 LocalSearch($G(V(G), E(G)), \epsilon$)

$S \leftarrow$ an arbitrary solution of G
$c \leftarrow$ a constant depending on ϵ
while there is a solution S' such that $|S \setminus S'| \leq c$, $|S' \setminus S| \leq c$ and $|S'| < |S|$ **do**
  $S \leftarrow S'$
output S

1.1 A brief overview of previous work

The first PTAS by Kleinberg and Kumar [22] considers two cases depending on the input graph, say G, has a highly connected pair or not. Intuitively, two vertices are highly connected if there are many vertex-disjoint paths between them[1]. If the input graph has no highly connected pairs, Kleinberg and Kumar show that the separator based algorithm by Lipton and Tarjan [24] is a PTAS. Otherwise, they consider two subcases: the region enclosed by the highly connected pair have a feedback vertex set of size at least or less than a constant $C = O(\epsilon^{-1})$. To dishtinguish between two subcases, they employ a fixed-parameter tractable algorithm by Downey and Fellows [12]. Then, in each subcase, they recursively approximate the FSV in subgrpahs of G. We note that there are several other smaller subcases in the algorithm.

In the algorithm by Demaine and Hajiaghayi [10], the devise a contraction decomposition framework to obtain a PTAS. Their framework use a constant approximation algorithm to guide the decomposition of the input graph into pieces where each piece has a feedback vertex set of constant size. By the bidimensional property of FVS, each piece has constant treewidth. Thus, they can use dynamic programming to optimally solve FVS problem in each piece. Then, the solutions of the pieces are glued together to form an approximate solution of the original problem. We note that subroutines used in the algorithm by Demaine and Hajiaghayi [10] are non-trivial. The recent algorithm by Cohen-Addad et al. [7] can be applied to vertex-weighted bounded-genus graphs but it is not simpler than the algorithm by Demaine and Hajiaghayi. In Section 4.1, we show a simple analysis of Algorithm 1 for bounded genus graphs using the matching bound from an old result by Nishizeki [26].

Fomin, Lokshtanov, Raman and Sauraubh [14] generalize Demaine and Hajiaghayi [10] result to minor-free graphs. Their algorithm, besides tools used by Demaine and Hajiaghayi, uses the deep theorem by Robertson and Seymour [30] in graph minor theory and is considered practically unimplementable [19]. In Section 4.3, we show that Algorithm 1 gives a PTAS for FVS problem in minor-free graphs. To simplify the presentation, we first give a proof for bounded-treewidth graphs in Section 4.2, that may be of independent interests. Our final proof is essentially a combination of two arguments for bounded genus graphs and bounded treewidth graphs. One drawback of our proof for minor-free graphs is that it uses the Robertson and Seymour theorem. We conjecture that it is possible to give a proof that

[1] See the paper by Kleinberg and Kumar [22] for the precise definition.

does not rely on the Robertson and Seymour theorem as other problems that are known to have local search PTAS in minor-free graphs.

1.2 Local Search Algorithms

Local search has been used before to obtain PTAS for other problems in minor-free graphs. Cabello and Gajser [6] gave local search PTAS for maximum independent set problem, minimum vertex cover problem and minimum dominating set problem in minor-free graphs. Cohen-Addad, Klein and Mathieu [8] showed that local search yields PTAS for k-means, k-median and uniform uncapacitated facility location in minor-free graphs. All their analysis relies on the *exchange graph*, a graph constructed from the optimal solution[2] O and the local search solution L. For independent set and vertex cover, the exchange graph is the subgraph induced by $\mathrm{O} \cup \mathrm{L}$, and for for the other problems, the exchange graph is built by contracting other vertices to vertices in the exchange graph. Then the local properties of these problems appear in the exchange graphs: if we consider a small neighborhood R in the exchange graph and replace the vertices of L in R with the vertices of O in R and the boundary of R, then the resulting vertex set is still a feasible solution for the original graph. Then by decomposing the exchange graph into small neighborhoods, we can bound the size of L by the size of O and all the boundaries of those neighborhoods.

However, FVS problem does not have such local property if we construct exchange graph only by deletion or only by contraction. This is because for a cycle C in the original graph, the vertex of L that covers C may be inside of some neighborhood but the vertex of O that covers C may be outside of that neighborhood. One may tries to argue the boundary of the neighborhood could cover C. But sadly, the boundary may not be helpful since the crossing vertices of C and the boundary may not be in both solutions and then they may be deleted or contracted to other vertices.

To solve this problem, we will construct an exchange graph with the following property: for any cycle C of the original graph, in our exchange graph there is either a vertex in $\mathrm{O} \cap \mathrm{L} \cap C$, or an edge between a vertex in $\mathrm{O} \cap C$ and a vertex in $\mathrm{L} \cap C$, or another cycle C' such that vertices in C' is a subset of vertices in C and $C' \cap (\mathrm{O} \cup \mathrm{L}) = C \cap (\mathrm{O} \cup \mathrm{L})$. To achieve this goal, we will apply both deletion and contraction to construct our exchange graph. When we delete some vertices, we may need to add some additional edges into our exchange graph. Further, we need to introduced vertices that are not in both solutions into the exchange graph. However, we need to guarantee that the number of added vertices is linear to the size of $\mathrm{O} \cup \mathrm{L}$; that is the main contribution of our work and makes our work different from all previous work.

We notice that if the input graph has bounded genus, then a simple bound for the independent high degree vertices implies the linear bound of the size of the exchange graph. This observation immediately implies the PTAS by local search for bounded genus graphs. However, the observation fails for graphs with bounded treewidth. For such graphs, we refine the tree decomposition and show that our construction can achieve the linear bound by an amortized argument. For minor-free graphs, we combine these two ideas based on the Roberson-Seymour decomposition theorem [30]. This theorem guarantees a tree decomposition where each bag in the tree decomposition contains a graph that can be almost embedded into a surface of constant genus. We then apply the first idea to bound the size of

[2] For k-means and k-median, the exchange graph is constructed from L and a nearly optimal solution O', which is obtained by removing some vertices of O.

each bag, and apply the second idea to refine the tree decomposition and bound the total size of the exchange graph.

To complement our positive results, we show two negative results for two variants of FVS problem, namely: *odd cycle transversal* and *subset feedback vertex set*. The odd cycle transversal (also called bipartization) problem asks, given an undirected graph, a minimum set of vertices whose removal results in a bipartite graph. The subset feedback vertex set problem asks, given an undirected graph and a subset U of vertices, a minimum set S of vertices such that after removing S the resulting graph contains no cycle that passes through any vertex of U. Details are presented in Section 5.

1.3 Other implications of our work

Showing that FVS problem in minor-free graphs has a simple local search PTAS is a first step toward a systematic characterization of those problems that admits local search PTASes. In planar graphs, FVS problem and connected dominating set (CDS) problem have played important roles as the motivation for new PTAS techniques. Baker's shifting technique [2], though very powerful, does not work for these two problems because of their non-locality. This motivates Demaine and Hajiaghayi [10] to develop the bidimensionality framework that unifies Baker's shifting technique and Lipton-Tarjan's separator approach. While more problems have been shown to admit local search PTASes, there is no single, unified characterization of such problems. We notice that local search PTAS for connected dominating set in low density graphs, which include minor-free graphs as a subclass, was shown by Har-Peled and Quanrud [17]. Our result can be seen as a complement of Har-Peled and Quanrud's work toward the better understanding of local search PTASes.

2 Preliminaries

For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. For a subgraph H of G, the *boundary* of H is the set of vertices that are in H but have at least one incident edge that is not in H. We denote by $int(H)$ the set of vertices of H that are not in the boundary of H. A graph H is a minor of G if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. G is H-minor-free if G does not contain a fixed graph H as a minor.

A *balanced separator* of a graph is a set of vertices whose removal partitions the graph roughly in half. In the seminal work of Lipton and Tarjan [23], they showed that planar graphs have a balanced separator of sublinear size. Then, their result is extended to bounded-genus graphs [11, 16, 21] and minor-free graphs [27, 4, 18, 28, 31]. An *r-division* is a decomposition of a graph, which is first introduced by Frederickson [15] for planar graphs.

▶ **Definition 1** (r-division)**.** For an integer r, an r-division of a graph G is a collection of edge-disjoint subgraphs of G, called *regions*, with the following properties.

1. Each region contains at most r vertices.
2. The number of regions is $O(n/r)$.
3. The number of boundary vertices, summed over all regions, is $O(n/\sqrt{r})$.

We say a graph is *r-divisible* if it has an r-division. Frederickson [15] gave a construction for the r-division of a planar graph which only relies on the separator theorem in planar graphs [23]. It is straightforward to extend the construction to any family of graphs with balanced separator of sublinear size. That implies:

▶ **Theorem 2** (Alon, Seymour, and Thomas [27] + Frederickson [15]). *Minor-free graphs are r-divisible.*

Since bounded genus graphs and bounded treewidth graphs are minor-free, they are r-divisible.

We borrow the following theorem from Nishizeki [26] to prove a bound for the size of an independent set of high degree vertices, which will be used in later analysis.

▶ **Theorem 3** (Nishizeki [26]). *Let G be a simple connected undirected graph with minimum degree three and smallest genus g that has at least ten vertices. Then the size of a maximum matching of G is at least $\lceil (n - 4g + 2)/3 \rceil$ where n is the number of vertices in G.*

▶ **Lemma 4.** *Let G be a connected graph of genus at most g that has at least two vertices. Let X be an independent set of G such that every vertex in X has degree at least 3 in G. Then $|X| \leq (2 + 4g)|V(G) \setminus X|$.*

Proof. Let $Y = V(G) \setminus X$. Consider a cellular embedding[3] of G on a surface Σ of genus at most g. We add edges, each of which is incident to at least one vertex in Y to G while maintaining the genus g, so that G is still simple and every vertex of Y has degree at least 3. This can be done by adding edges inside faces of length at least 4 of the cellular embedding of G on Σ. Call the resulting graph G'. By Theorem 3, G' has a matching M such that

$$|M| \geq (n - 4g + 2)/3 \tag{1}$$

where n is the number of vertices of G' (also the number of vertices of G). Since each added edge is incident to at least one vertex in Y, X is still an independent set in G'. Thus, each edge in M is incident to at least one vertex in Y and $|Y| \geq |M|$. By Equation (1), we have:

$$|Y| \geq (|Y| + |X| - 4g + 2)/3$$

That implies:

$$|X| \leq 2|Y| + 4g - 2 \leq 2|Y| + 4g \leq (2 + 4g)|Y|$$

◀

3 Exchange graph implies PTAS by Local search

In this section, we show that if for an H-minor-free graph G we can construct another graph, called *exchange graph*, such that it is r-divisible, then Algorithm 1 is a PTAS for FVS in G. In our analysis below, we choose the constant $c = O_H(1/\epsilon^2)$ in Algorithm 1, where $O_H(\cdot)$ notation hides the factors depending on the size of the minor H.

Let O be an optimal solution of the FVS problem and L be the output of the local search algorithm.

▶ **Definition 5.** A graph Ex is an exchange graph for the optimal solution O and the local solution L of FVS problem in a graph G if it satisfies the following properties:

(1) $\mathrm{L} \cup \mathrm{O} \subseteq V(\mathrm{Ex}) \subseteq V(G)$.
(2) $|V(\mathrm{Ex})| = O(|\mathrm{L}| + |\mathrm{O}|)$.

[3] A graph is *cellularly embedded* if its faces are homeomorphic to open disks.

(3) For every cycle C of G, there is (a) a vertex of C in $\mathrm{O} \cap \mathrm{L}$ or (b) an edge $uv \in E(\mathrm{Ex})$ between a vertex $u \in \mathrm{L}$ and a vertex $v \in \mathrm{O}$ in C or (c) a cycle C' of Ex such that $V(C') \subseteq V(C)$ and $C \cap (\mathrm{O} \cup \mathrm{L}) = C' \cap (\mathrm{O} \cup \mathrm{L})$.

▶ **Theorem 6.** *If graph G has an r-divisible exchange graph for an optimal solution* O *and a local solution* L*, then Algorithm 1 is a polynomial-time approximation scheme for feedback vertex set problem in G, whose running time is $n^{O_H(1/\epsilon^2)}$ where n is the number of vertices in G.*

Proof. Let Ex be an r-divisible exchange graph for O and L. Since Ex is r-divisible, we can find an r-division of Ex for $r = 1/\delta^2$, where δ depends on ϵ and we decide later. Let B be the multi-set containing all the boundary vertices in the r-division. By the third property of r-division, $|B|$ is bounded by $O(|V(\mathrm{Ex})|/\sqrt{r})$. By the second property of exchange graph, $|V(\mathrm{Ex})|$ is at most $O(|\mathrm{O}| + |\mathrm{L}|)$. Then we have for some constant c_1

$$|B| \leq c_1\delta(|\mathrm{O}| + |\mathrm{L}|) \tag{2}$$

If we can prove the difference between the two solutions is bounded by constant times of the size of B, that is for some constant c_2

$$|\mathrm{L}| \leq |\mathrm{O}| + c_2|B|, \tag{3}$$

then by setting $\delta = \epsilon/(2c_1c_2 + c_1c_2\epsilon)$, we have $|\mathrm{L}| \leq (1+\epsilon)|\mathrm{O}|$, giving the approximation ratio.

To prove Equation (3), we need the properties of Ex. For any region R_i of the r-division, let B_i be the boundary of R_i and M_i be the union of $\mathrm{L} \setminus R_i$, $\mathrm{O} \cap R_i$ and B_i.

▶ Claim 7. M_i is a feedback vertex set of G.

Proof. For a contradiction, assume there is a cycle C of G that is not covered by M_i. Then C does not contain any vertex of $\mathrm{L} \setminus R_i$, $\mathrm{O} \cap R_i$ and B_i. So C can only be covered by some vertices of $(\mathrm{L} \setminus \mathrm{O}) \cap int(R_i)$ and some vertices of $\mathrm{O} \setminus (\mathrm{L} \cup R_i)$. This implies that C does not contain any vertex of $\mathrm{O} \cap \mathrm{L}$ and there is no edge in Ex between $C \cap \mathrm{O}$ and $C \cap \mathrm{L}$. By the third property of exchange graph, there must be a cycle C' in Ex such that $V(C') \subseteq V(C)$ and $C \cap (\mathrm{O} \cup \mathrm{L}) = C' \cap (\mathrm{O} \cup \mathrm{L})$. Let u be the vertex of $(\mathrm{L} \setminus \mathrm{O}) \cap int(R_i)$ in C and v be the vertex of $\mathrm{O} \setminus (\mathrm{L} \cup R_i)$ in C. Then cycle C' contains both u and v, which implies C' crosses the boundary of R_i, that is $C' \cap B_i \neq \emptyset$. Let w be a vertex in $C' \cap B_i$, then w also belongs to C in G. This implies M_i contains a vertex of C, a contradiction. ◀

By the construction of M_i, we know the difference between L and M_i is bounded by the size of the region R_i, that is r. Since L is the output of Algorithm 1, we know L cannot be improved by changing at most $c = r$ vertices. So we have $|\mathrm{L}| \leq |M_i|$. By the construction of M_i, this implies

$$|\mathrm{L} \cap R_i| \leq |M_i \cap R_i| \leq |\mathrm{O} \cap int(R_i)| + |B_i|.$$

By this equation, we have

$$|\mathrm{L} \cap int(R_i)| \leq |\mathrm{L} \cap R_i| \leq |\mathrm{O} \cap int(R_i)| + |B_i| \leq |\mathrm{O} \cap R_i| + |B_i|.$$

Summing over all regions in the r-division, we can have

$$|\mathrm{L}| - |B| \leq \sum_i |\mathrm{L} \cap int(R_i)| \leq \sum_i (|\mathrm{O} \cap R_i| + |B_i|) \leq |\mathrm{O}| + 2|B|.$$

This proves Equation (3).

Now we analyze the running time of Algorithm 1. In each iteration, the algorithm tries to improve the current solution, so the size of the solution increases by at least one. Thus, the number of iterations is $O(n)$. Since each iteration needs $O(n^c)$ time for some constant $c = O_H(1/\epsilon^2)$, the total running time is $n^{O_H(1/\epsilon^2)}$. ◀

4 Exchange graph construction

By Theorem 6, it remains to show that there is an r-divisible exchange graph for any H-minor-free graph G. We say a vertex is a *solution vertex* if it is in $\mathrm{O} \cup \mathrm{L}$. To construct the exchange graph for G, we first delete all edges that are incident to vertices of $\mathrm{O} \cap \mathrm{L}$, and remove all components that do not contain any solution vertex. Note that the removed components are acyclic. Then we contract edges that have an endpoint that is not a solution vertex and has degree at most two until there is no such edge in the resulting graph. Let K be the resulting graph. We say a vertex is a *Steiner vertex* if it is not a solution vertex in K.

Since L and O are both feedback vertex set of G, every cycle of K must contain a vertex from L and a vertex from O. Since edges incident to vertices of $\mathrm{O} \cap \mathrm{L}$ are removed, K has no self-loop. However, K could have parallel edges and parallel edges must be between two vertices in different solutions. We keep K simple by further removing parallel edges. Since we only remove edges between solution vertices, we have:

▶ Observation 8. Every Steiner vertex of K has degree at least 3.

Since $\mathrm{O} \cup \mathrm{L}$ is a feedback vertex set of K, $K \setminus (\mathrm{O} \cup \mathrm{L})$ is a forest F containing only Steiner vertices. For each tree T in F, we define the *degree* of T, denoted by $\deg_K(T)$, as the number of edges in K between T and $\mathrm{O} \cup \mathrm{L}$. Let $\ell(T)$ be the number of leaves of T. By Observation 8, every internal vertex of T has degree at least 3. Thus, $|V(T)| \leq 2\ell(T)$. That implies:

$$|V(T)| \leq 2\deg_K(T). \tag{4}$$

We observe that graph K satisfies the first and third properties in Definition 5 before we remove parallel edges. Since all parallel edges are between solution vertices, K satisfies these two properties after we remove all parallel edges. In Section 4.1, we show that if G has bounded genus, K is an r-divisible exchange graph. However, when G only has bounded treewidth, we need to further modify K to guarantee it is an r-divisible exchange graph.

4.1 Exchange graph for bounded genus graphs

In this section, we prove the following theorem.

▶ **Lemma 9.** *If G has genus g, then we have $|V(K)| \leq (19 + 36g)(|\mathrm{L} \cup \mathrm{O}|)$.*

Proof. Since each vertex in $\mathrm{O} \cap \mathrm{L}$ is isolated in K, every non-trivial connected component of K does not contain any vertex of $\mathrm{O} \cap \mathrm{L}$. In the argument below, we assume that K is connected since otherwise we can apply the argument for each component of K separately. We contract each tree T of F into a single Steiner vertex s_T. Let K' be the resulting graph. We have:

▶ Observation 10. Graph K' is simple.

Proof. Since every cycle of K must contains a vertex from L and a vertex from O, there cannot be any solution vertex in K that is incident to more than one vertex of a tree T of F. So there cannot be parallel edges in K'. ◀

Let X be the set of Steiner vertices of K'. By the construction of K', set X is an independent set of K'. By Observation 8, every vertex of X has degree at least 3. Since K' is a minor of G and since G has genus g, graph K' has genus at most g. By Lemma 4, $|X| \leq (2+4g)|\mathrm{O} \cup \mathrm{L}|$. Thus, $|V(K')| \leq (3+4g)|\mathrm{O} \cup \mathrm{L}|$.

By Euler-Poincare formula, we have:

$$\begin{aligned} |E(K')| &\leq 3|V(K')| + 6g - 6 \\ &\leq 3(3+4g)|\mathrm{O} \cup \mathrm{L}| + 6g - 6 \\ &\leq (9+18g)|\mathrm{O} \cup \mathrm{L}| \end{aligned} \tag{5}$$

We have:

$$\begin{aligned} |V(K) \setminus (\mathrm{O} \cup \mathrm{L})| &= \textstyle\sum_{T \in F} |V(T)| \\ &\leq 2 \textstyle\sum_{T \in F} \deg_K(T) && \text{(Equation (4))} \\ &= 2 \textstyle\sum_{T \in F} \deg_{K'}(s_T) \\ &\leq 2|E(K')| && (\{s_T | T \in F\} \text{ is an independent set}) \\ &\leq 2(9+18g)|\mathrm{O} \cup \mathrm{L}| && \text{(Equation (5))} \end{aligned}$$

This implies the lemma. ◀

Since K is a minor of G and G has bounded genus, graph K also has bounded genus. Thus, by Lemma 9, we have:

▶ **Corollary 11.** *Algorithm 1 is a PTAS for Feedback Vertex Set problem in bounded genus graphs.*

4.2 Exchange graph for bounded treewidth graphs

Before presenting the full construction, we review some fundamental concepts related to bounded treewidth graphs.

▶ **Definition 12** (Tree decomposition). A *tree decomposition* of G is a pair $(\mathcal{T}, \mathcal{X})$ where $\mathcal{T}$ is a tree, and $\mathcal{X} = \{X_i | i \in V(\mathcal{T})\}$ is a family of subsets of $V(G)$ such that

1. the union of all sets X_i is $V(G)$;
2. for each edge $uv \in E(G)$, there is a bag X_i containing both u and v;
3. for any vertex $v \in V(G)$, the set of nodes $\{i \in V(\mathcal{T}) | v \in X_i\}$ forms a subtree of $\mathcal{T}$.

To distinguish the vertices in the original graph G and vertices of $\mathcal{T}$, we call vertices of $\mathcal{T}$ *nodes* and their corresponding X_i *bags*. The *width* of a tree decomposition is $\max_{i \in V(\mathcal{T})} |X_i| - 1$ and the *treewidth* of G is the minimum width among all possible tree decompositions of G.

▶ **Lemma 13** (Bodlaender [5]). *If G has treewidth k, then $|E(G)| \leq k|V(G)| - k(k+1)/2$.*

In Subsection 4.1, we apply Lemma 4 to bound the number of Steiner vertices in graph K. However, Lemma 4 does not hold for bounded treewidth graph. A counterexample could be the complete bipartite graph $K_{3,\ell}$: it has treewidth at most four and each vertex in the big part has degree three, but the number of such vertices is unbounded. Therefore, we need a different idea to construct the exchange graph for the bounded treewidth graphs. Recall $F = K \setminus (\mathrm{O} \cup \mathrm{L})$ is a forest of Steiner vertices. Our idea is based on the following observation.

▶ Observation 14. Let T be a tree in F and N_T be the set of neighbors of leaves of T in $\mathrm{O} \cup \mathrm{L}$. If we remove T and adding edges to K so that N_T induces a clique in K, then K still satisfies properties (1) and (3) in Definition 5.

Ideally, we would like to apply the modification in Observation 14 to all Steiner vertices, which may give us an exchange graph. However, this graph may not be r-divisible since its treewidth may be very large. Therefore, we need to balance the treewidth and the number of Steiner vertices so that the resulting graph satisfies r-divisibility and the properties of exchange graph at the same time. In the following, we show that if we carefully choose the trees in F, which do not have many neighbors in K, to apply the modification in Observation 14, then we can obtain an exchange graph with bounded treewidth from K.

W.l.o.g. we assume K is connected, since otherwise we can apply the argument for each component of K separately. Let k be the treewidth of G. Since graph K is a minor of G, we know K has treewidth at most k. Let F_0 be the set of trees in F whose degree is at least $k+1$. By the following lemma, we can bound the total size of F_0.

▶ **Lemma 15.** *The total size of F_0 is at most $2k(k+1)|\mathrm{O} \cup \mathrm{L}|$.*

Proof. Let U be the graph obtained from K by contracting each tree T of F into a single Steiner vertex s_T. Since L and O are both feedback vertex sets, each cycle of K contains at least one vertex from O and one vertex from L. Thus, $\deg_U(s_T) = \deg_K(T)$.

Let X be the set of Steiner vertices of U that have degree at least $k+1$. Consider the subgraph $U[X \cup \mathrm{L} \cup \mathrm{O}]$ of U induces by $X \cup \mathrm{L} \cup \mathrm{O}$. Since K has treewidth k, subgraph $U[X \cup \mathrm{L} \cup \mathrm{O}]$ has treewidth at most k. Because X is an independent set, we have:

$$\begin{aligned}(k+1)|X| &\leq |E(U[X \cup \mathrm{L} \cup \mathrm{O}])| \\ &\leq k(|X| + |\mathrm{O} \cup \mathrm{L}|) \qquad \text{(Lemma 13)}\end{aligned}$$

Thus, $|X| \leq k|\mathrm{O} \cup \mathrm{L}|$. Hence, $|E(U[X \cup \mathrm{L} \cup \mathrm{O}])| \leq k(k+1)|\mathrm{O} \cup \mathrm{L}|$. We have:

$$\begin{aligned}\textstyle\sum_{T \in F_0} |V(T)| &\leq \textstyle\sum_{T \in F_0} 2\deg_K(T) && \text{(Equation (4))} \\ &= 2\textstyle\sum_{T \in F_0} \deg_U(s_T) \\ &\leq 2|E(U[X \cup \mathrm{L} \cup \mathrm{O}])| && (\{s_T | T \in F\} \text{ is an independent set}) \\ &\leq 2k(k+1)|\mathrm{O} \cup \mathrm{L}|\end{aligned}$$

◀

Let $S = \mathrm{O} \cup \mathrm{L} \cup V(F_0)$. By Lemma 15, we have

$$|S| \leq (2k^2 + 2k + 1)|\mathrm{O} \cup \mathrm{L}| \tag{6}$$

Let U be the graph obtained from K by contracting every tree of $F \setminus F_0$ into a single Steiner vertex. For now, we only consider vertices in $V(U) \setminus S$ to be Steiner vertices and every vertex of S is called *non-Steiner*. Graph U has treewidth at most k since it is a minor of K. Let $(\mathcal{T}, \mathcal{X})$ be a tree decomposition of width at most k for U. We say $(\mathcal{T}, \mathcal{X})$ is a *succinct* tree decomposition if:

(i) For every two adjacent nodes i and j in $\mathcal{T}$, the two corresponding bags X_i and X_j satisfy that $X_i \not\subseteq X_j$ and $X_j \not\subseteq X_i$.

(ii) For every vertex v, let $\mathcal{T}_v$ be the subtree of $\mathcal{T}$ consisting of all nodes whose corresponding bags contain v. For every leaf i of $\mathcal{T}_v$, the vertex v must have at least one neighbor in X_i that is not in any other bags of $\mathcal{T}_v$.

From any tree decomposition $(\mathcal{T}, \mathcal{X})$, we can make it succinct by repeatedly applying the two following operations: contracting edge ij of the tree $\mathcal{T}$ if the corresponding bags X_i and X_j violate property (i) and removing v from the bag corresponding to any leaf of $\mathcal{T}_v$ that violates property (ii). This implies the following observation.

▶ Observation 16. We can make $(\mathcal{T}, \mathcal{X})$ succinct while keeping the underlying graph unchanged.

Recall S is the union of O, L and F_0. Now we introduce S-*succinct* tree decomposition. We first root $\mathcal{T}$ at a non-leaf node r. Let α be a node in $\mathcal{T}$ and $\beta_1, \beta_2, \ldots, \beta_j$ be the children of α. If α is a leaf node, then $j = 0$. We call vertices in $X_\alpha \setminus (\cup_{i=1}^{j} X_{\beta_i})$ *introduced vertices* of α. This implies that if α is a leaf node, then every vertex of X_α is an introduced vertex. We call a vertex v in X_α a *forget vertex* if v is not in the bag corresponding to the parent of α. We regard every vertex in the bag X_r corresponding to root node r as a forget vertex. We call a vertex v *dangling* if it is both an introduced vertex and a forget vertex of α. We note that a vertex v could be a forget vertex of only one node but it can be introduced in multiple nodes. We say $(\mathcal{T}, \mathcal{X})$ is S-*succinct* if:

1. $(\mathcal{T}, \mathcal{X})$ is succinct.
2. There is no dangling Steiner vertex in $(\mathcal{T}, \mathcal{X})$.

We now transform a succinct tree decomposition $(\mathcal{T}, \mathcal{X})$ into S-succinct. Our transformation affects both graphs U and K. Suppose that v is a dangling Steiner vertex of a node α in $\mathcal{T}$. For graph U, we remove v from U and the tree decomposition $(\mathcal{T}, \mathcal{X})$. We then add edges between neighbors of v in U so that neighbors of v induce a clique in the resulting graph. Since v is a dangling vertex, every neighbor of v is in X_α. Thus, adding edges between neighbors of v still preserve the width of the tree decomposition. We note that the removing v could make X_α become a subset of another bag X_β for β being a neighbor of α in $\mathcal{T}$. In this case, we contract X_α to X_β. For graph K, we remove all vertices of the tree corresponding to v in F, and add edges between neighbors of T in the resulting graph so that those neighbors induce a clique. By Observation 14, properties (1) and (3) in Definition 5 are preserved. We apply this modification to the resulting graph and its tree decomposition until the resulting tree decomposition is S-succinct. Next lemma shows that an S-succinct tree decomposition implies that we can bound the size of U by the size of S.

▶ **Lemma 17.** *If graph U has an S-succinct tree decomposition $(\mathcal{T}, \mathcal{X})$ of width k, then $|V(U)| \leq (4k^2 + 8k + 5)|S|$.*

Proof. To prove this lemma, we will apply an amortized argument to bound the size of U. That is, we will assign each Steiner vertex to a non-Steiner vertex (the vertex in S) such that there are at most $4(k+1)^2$ different Steiner vertices assigned to the same non-Steiner vertex. Then the lemma follows.

We collect non-Steiner vertices into a set C during a post-order traversal of the tree $\mathcal{T}$. Initially, $C = \emptyset$. During the collection, we assign Steiner vertices to non-Steiner vertices in C and may mark some leaf nodes of $\mathcal{T}$ as *unavailable*. Initially, leaf nodes of $\mathcal{T}$ are all marked *available*. During the traversal, we would maintain the following invariant:

> **Marking invariant:** For each non-leaf node α whose parent is not visited, there is at least one available leaf of $\mathcal{T}$ that is a descendant of α.

For a node α, we denote by $\mathcal{T}[\alpha]$ the subtree of $\mathcal{T}$ rooted at α and by $\mathcal{X}[\alpha]$ the union of bags corresponding to nodes in $\mathcal{T}[\alpha]$. Let α be the node of $\mathcal{T}$ that we are currently visiting. We have three cases depending on the number of children of α: zero, one or at least two. If α is a leaf node or has at least two children, we would show that every Steiner vertex in $\mathcal{X}[\alpha]$ is assigned to a non-Steiner vertex in C. However, if α has only one child, then there will be a situation where we do not immediately assign the Steiner vertices in $\mathcal{X}[\alpha]$ but delay the assignment.

Case 1: node α is a leaf. Let $Y = X_\alpha \cap S$. Since $(\mathcal{T}, \mathcal{X})$ is S-succinct, $Y \neq \emptyset$ and every forget vertex of α is in S. We add vertices of Y into C, and assign all Steiner vertices in X_α to one vertex in Y. Since $|X_\alpha| \leq k+1$, there are at most k Steiner vertices assigned to that vertex in Y.

Case 2: node α has exactly one child in $\mathcal{T}$. Let L_2 be the set of unassigned Steiner vertices in $\mathcal{X}[\alpha]$. Note that there could be unassigned Steiner vertices in X_β for some descendant β of α. For this case, we have two subcases.

1. If X_α contains a vertex $v \in S$ that is not in C or is a forget vertex of X_α, we add v into C if $v \notin C$ and assign all vertices in L_2 to v. If there are other vertices of X_α in S that are currently not in C, we add those into C as well.
2. Otherwise, we do nothing and visit the next bag. In this case, we call α a *skipped node*.

The marking invariant holds at α inductively since we do not mark any leaf node of $\mathcal{T}$ in this case.

Case 3: node α has at least two children in $\mathcal{T}$. Let $\beta_1, \beta_2, \ldots, \beta_p$ be the children of α. Recall that, by the marking invariant, each subtree $\mathcal{T}[\beta_i]$ has at least one available leaf node. Let γ_i be an available leaf node in $\mathcal{T}[\beta_i]$ for $1 \leq i \leq p-1$. Let v_i be a forget vertex of γ_i in S for $1 \leq i \leq p-1$. Such vertex v_i exists since $\mathcal{T}$ is S-succinct and γ_i is a leaf node. Note that v_i is non-Steiner. Let L_3 be the set of unassigned Steiner vertices in $\mathcal{X}[\alpha]$. We partition vertices of L_3 into $p-1$ groups, say $Z_1, Z_2, \ldots, Z_{p-1}$, so that their sizes differ by at most one. We assign each group Z_i to the vertex v_i. Finally, we mark γ_i unavailable for $1 \leq i \leq p-1$. The marking invariant holds since α has at least two children. We also add vertices of $X_\alpha \cap S$ into C.

By the above three cases, we have:

▶ Observation 18. If α is not a skipped node, then every Steiner vertices in $\mathcal{X}[\alpha]$ is assigned to some non-Steiner vertex in C after we visit α.

We first bound the number of unassigned Steiner vertices in Case 2.

▶ Claim 19. For Case 2, we have $|L_2| \leq (k+1)^2$ for any node α.

Proof. Since $|L_2 \cap X_\alpha| \leq |X_\alpha| \leq k+1$, we only need to bound $|L_2 \setminus X_\alpha|$. To achieve this, we will show that we can map each vertex in $L_2 \setminus X_\alpha$ to a vertex in $S \cap X_\alpha$ such that there are at most k different vertices that are mapped to the same vertex in $S \cap X_\alpha$. Since $|X_\alpha| \leq k+1$, we have $|L_2| \leq |L_2 \cap X_\alpha| + |L_2 \setminus X_\alpha| \leq (k+1) + k|S \cap X_\alpha| \leq (k+1) + k(k+1) = (k+1)^2$.

Let v be a vertex in $L_2 \setminus X_\alpha$ and v^* be a node of $\mathcal{T}$ such that v is a forget vertex of v^*. We will map v to a vertex u in $S \cap X_\alpha$. Let β be the child of α. By Observation 18, nodes in the subpath of $\mathcal{T}$ between v^* and β are skipped. So node v^* has degree two. Since v is a forget vertex of node v^*, we know that v^* must be a leaf node of $\mathcal{T}_v$, the subtree of $\mathcal{T}$ consisting of all the nodes whose corresponding bags contain v. By condition (ii) of succinctness, vertex v must have a neighbor in S that is an introduced vertex of node v^*. Let u be such a neighbor of v. Since v^* is a skipped node, we know that u is in C before we visit v^*. Thus, vertex u must be an introduced vertex of another node, say u^*, that we visit before visiting v^* in the post-order traversal of $\mathcal{T}$. This implies $u^* \notin \mathcal{T}[v^*]$. Since nodes in the subpath between v^* and α all have degree two, node u^* cannot be a descendant of α. By the third condition of tree decomposition (Definition 12), vertex u must be in X_α. Also since the nodes in the subpath between v^* and α in $\mathcal{T}$ have degree two, vertex u can only be introduced once in the subtree $\mathcal{T}[X]$, which means we only map forget vertices of v^* to u.

Since $|X_{v^*}| \leq k+1$ and $u \in X_{v^*}$, we have $|L_2 \cap X_{v^*}| \leq k$, which implies there are at most k vertices that could be mapped to u. ◀

▶ Claim 20. For Case 3, we have $|L_3| \leq (pk+1)(k+1)$ for any node α with p children.

Proof. Let β_i be the children of α for $1 \leq i \leq p$. Since $(\mathcal{T}, \mathcal{X})$ has width at most k, we know $|L_3 \cap X_\alpha| \leq k+1$. By the same argument as that for Claim 19, we can show that there are at most $k(k+1)$ vertices in L_3 that belong to $\mathcal{X}[\beta_i]$ for each $1 \leq i \leq p$. And then the claim follows. ◀

Now we bound the number of Steiner vertices assigned to any vertex in C. We observe that each time a non-Steiner vertex v is added in Case 1 or Case 3, there are at most k Steiner vertices assigned to v. By Claim 19, there are at most $(k+1)^2$ Steiner vertices assigned to one non-Steiner vertex in Case 2. Further, we only assign more vertices to a non-Steiner vertex v in two situations: (a) when we visit a node α in Case 2 and vertex v is its forget vertex and (b) when we visit a node α in Case 3 and vertex v is a forget vertex of an available leaf node in $\mathcal{T}[\alpha]$. In the former case, by Claim 19, we assign to v at most $(k+1)^2$ more vertices. In the latter case, the number of additional vertices we assign to v is at most:

$$\left\lceil \frac{(pk+1)(k+1)}{p-1} \right\rceil \leq 2(k+1)^2 \quad (p \geq 2)$$

Thus, each vertex in C is assigned at most $4(k+1)^2$ Steiner vertices. ◀

Let K' be obtained from U by uncontracting each Steiner vertex in U (of degree at most k) to a tree in the forest F. We can bound the size and treewidth of K' by the following lemma.

▶ **Lemma 21.** *The size of K' is at most $O(k^5|\mathrm{O} \cup \mathrm{L}|)$ and the treewdith of K' is at most $O(k^2)$.*

Proof. By Equation (6) and Lemma 17, we have:

$$|V(U)| \leq (4k^2 + 8k + 5)(2k^2 + 2k + 1)|\mathrm{O} \cup \mathrm{L}|$$

Since T has degree at most k, by Equation (4), $|V(T)| \leq 2k$. Thus, we have:

$$|V(K')| \leq 2k(4k^2 + 8k + 5)(2k^2 + 2k + 1)|\mathrm{O} \cup \mathrm{L}|$$

Given a tree decomposition of width k of U, we replace each Steiner vertex s_T by at most $2k$ vertices of $V(T)$. Therefore, the resulting decomposition is a tree decomposition of K' whose width is at most $2k(k+1)$. ◀

By Lemma 21, graph K' is an exchange graph and has bounded treewidth. Thus, K' is r-divisible and we have:

▶ **Corollary 22.** *Algorithm 1 is a PTAS for Feedback Vertex Set problem in bounded treewidth graphs.*

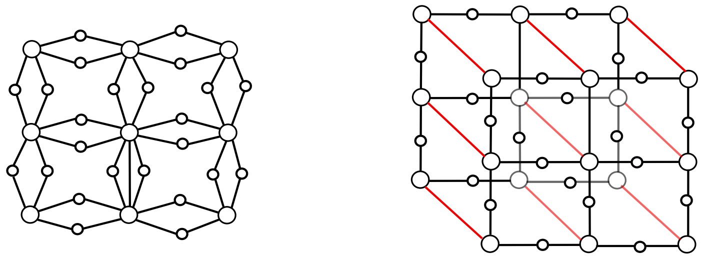

Figure 1 An example shows that a graph obtained by uncontracting a matching from a planar graph can have a big minor. Starting from an $n \times n$ grid, we double every edge and subdivide parallel edges to make the graph simple (figure (a)). We then uncontract a matching from the original grid vertices (figure (b)). The resulting graph has K_h minor where $h = \Omega(n)$. This example is adapted from [9].

4.3 Exchange graph for minor-free graphs

We combine two ideas that we developed in Section 4.1 and Section 4.2 to construct an r-divisible exchange graph when input graph G is minor-free. We assume that graph K is connected since we can apply our argument to each component of K separately. We will remove Steiner vertices from K while preserving properties (1) and (3) in Definition 5 until $|V(K)| = O(|\mathrm{L} \cup \mathrm{O}|)$. Our main tool is Observation 14.

We describe the high level construction here. We start with K and keep the trees of high degree as we did in Section 4.2. For low-degree trees of Steiner vertices, we contract each tree into a single Steiner vertex. Let U be the contracted graph. We then remove some Steiner vertices from U and the corresponding tree from K. To choose which Steiner vertex to remove, we apply the Robertson-Seymour decomposition theorem [30] to find a tree decomposition of U where each bag of the tree can be almost embedded into a surface of constant genus. If one bag in the decomposition is too big, then we will further contract Steiner vertices in the surface embedded parts of the graph so that the number of Steiner vertices can be bounded by a constant times of the number of non-Steiner vertices by Lemma 4. This is similar to the idea in Section 4.1. After that, we have a tree decomposition such that each bag is not very big. We then apply the idea in Section 4.2 to remove Steiner vertices. Finally, we uncontract Steiner vertices in the resulting graph. However, different from Section 4.2, where we can argue that the exchange graph has bounded treewidth (Lemma 21), the uncontracted graph could have big minors. In Figure 1, we show an example where we uncontract a matching from a planar graph of $O(n^2)$ vertices, we obtain a graph with K_n minor. Fortunately, we are still able to argue that the uncontracted graph is r-divisible.

4.3.1 Robertson-Seymour Decomposition Theorem

Let $(\mathcal{T}, \mathcal{X})$ be a tree decomposition of G. Let α and β be two adjacent nodes in $\mathcal{T}$. We define $E(\alpha, \beta)$ to be the set of edges of the clique with vertex set $X_\alpha \cap X_\beta$. A *torso* of $(\mathcal{T}, \mathcal{X})$ is a graph H_α such that $V(H_\alpha) = X_\alpha$ for some node α in $\mathcal{T}$, and $E(H_\alpha) = E(G[X_\alpha]) \cup E(\alpha, \beta_1) \ldots \cup E(\alpha, \beta_p)$ where $\beta_1, \beta_2, \ldots, \beta_p$ are neighbors of α in $\mathcal{T}$. We call edges in $E(H_\alpha) \setminus E(G[X_\alpha])$ *virtual edges*. The definition of torso implies that $X_\alpha \cap X_{\beta_i}$, for any $1 \le i \le p$, induces a clique in H_α. We call cliques $H_\alpha[X_\alpha \cap X_{\beta_1}], H_\alpha[X_\alpha \cap X_{\beta_2}], \ldots, H_\alpha[X_\alpha \cap X_{\beta_p}]$ *legs* of the torso H_α. We note that legs of a torso may not be vertex-disjoint.

▶ **Definition 23** (h-nearly-embeddable graph)**.** A graph G is *h-nearly-embeddable* in a surface

Σ with h boundary cycles $C_1, C_2, \ldots, C_h$ if there exists a subset of vertices, say A, and $h+1$ (possibly empty) subgraphs $G_0, G_1, \ldots, G_h$ of G such that:

1. $|A| \le h$.
2. G_0 is embeddable in a surface Σ of genus at most h.
3. $V(G_0 \cup G_1 \cup \ldots \cup G_h) = V(G) \setminus A$.
4. $G_1, G_2, \ldots, G_h$ are pairwise vertex-disjoint.
5. For each $1 \le i \le h$, subgraph G_i has a width-h path decomposition with ℓ bags $B_{i1}, B_{i2}, \ldots, B_{i\ell}$ such that:
   a. There are ℓ consecutive vertices $v_{i1}, v_{i2}, \ldots, v_{i\ell}$ ordered clock-wisely along C_i such that $v_{ij} \in B_{ij}$ for all $1 \le j \le \ell$.
   b. $V(G_0) \cap V(G_i) = \{v_{i1}, \ldots, v_{i\ell}\}$.

We call A the set of *apices* of G and call subgraphs $G_1, \ldots, G_h$ *vortices* of G. We denote the h-RS-nearly-embeddable decomposition of G by $(A, G_0, G_1, \ldots, G_h)$.

▶ **Definition 24** (h-RS-decomposability)**.** A graph G is *h-RS-decomposable* if G has a tree decomposition $(\mathcal{T}, \mathcal{X})$ whose torsos are h-nearly-embeddable graphs.

We call such $(\mathcal{T}, \mathcal{X})$ an *h-RS-tree decomposition* of G.

▶ **Theorem 25** (Robertson and Seymour [30])**.** *If a graph G is H-minor-free, it is h-RS-decomposable where h is a constant that only depends on $|V(H)|$.*

We note that in an h-RS-tree decomposition, any leg of any torso has size at most h since the torso is h-nearly-embeddable and since the leg is a clique.

▶ **Lemma 26.** *If a graph G is h-RS-decomposable, there is a constant h' that only depends on h such that G excludes $K_{h'}$ as a minor.*

Proof. Let $h' = \max\{38, 2h\}$. For a contradiction, assume $K_{h'}$ is a minor of G. Since h-RS-decomposability is a minor-closed property, graph $K_{h'}$ has an h'-RS-tree decomposition $(\mathcal{T}, \mathcal{X})$. It can be proved by the definition of tree decomposition that each bag in the tree decomposition of a complete graph contains all vertices in the complete graph. Then any torso in the decomposition $(\mathcal{T}, \mathcal{X})$ is a complete graph $K_{h'}$. Ringel and Youngs [29] proved that for all $n \ge 3$, the genus of the complete graph K_n is $\lceil (n-3)(n-4)/12 \rceil$. So $K_{h'}$ cannot be h-nearly-embeddable for $h \ge 19$, and the decomposition $(\mathcal{T}, \mathcal{X})$ is not an h-RS-tree decomposition, a contradiction. ◀

4.3.2 Exchange graph construction

Since K is a minor of G, it is H-minor-free. Recall that an H-minor-free of n vertices has at most $c_0\sigma_H$ edges where c_0 is a constant independent of H and $\sigma_H = |V(H)|\sqrt{\log |V(H)|}$ (see Mader [25]). Also, recall F is a forest of Steiner vertices obtained by removing $\mathrm{O} \cup \mathrm{L}$ from K. Let F_0 be the set of trees of degree at least $c_0\sigma_H + 1$ in F. Then we can bound the size of F_0 by the following lemma.

▶ Claim 27. The size of F_0 is at most $2c_0\sigma_H(c_0\sigma_H + 1)|\mathrm{O} \cup \mathrm{L}|$.

Proof. The proof follows exactly the proof of Lemma 15 by replacing k with $c_0\sigma_H$. ◀

Let S be the union of O, L and $V(F_0)$, that is

$$S = \mathrm{O} \cup \mathrm{L} \cup V(F_0).$$

By Claim 27, there is a constant c_1 that only depends on $|V(H)|$ such that:

$$|S| \leq c_1|\mathrm{O} \cup \mathrm{L}| \tag{7}$$

Let U be the graph obtained from K by contracting every tree of $F \setminus F_0$ into a single Steiner vertex. Herein, we call vertices of $V(U) \setminus S$ Steiner vertices and call vertices of S non-Steiner. Since U is a minor of K, it is H-minor-free. By Theorem 25, there is an h-RS-tree decomposition $(\mathcal{T}, \mathcal{X})$ of U, where h only depends on $|V(H)|$. We root $(\mathcal{T}, \mathcal{X})$ at a leaf node. The notion of forget, introduced and dangling vertices are carried from Section 4.2. We will refine the tree decomposition $\mathcal{T}$ in two phases to obtain a canonical h_0-RS-tree decomposition where $h_0 = 2h$. We define an operation $\Delta(v)$ for a Steiner vertex v in U as follows: remove v and then add edges between its neighbors so that all the neighbors induce a clique and the resulting graph is simple. During the processing, we maintain the properties (1) and (3) in Definition 5 by the following lemma.

▶ **Lemma 28.** *Assume U satisfies properties (1) and (3) in Definition 5. Let α be a node in $\mathcal{T}$, and let v be a dangling Steiner vertex of α. After applying the operation $\Delta(v)$, the resulting graph satisfies properties (1) and (3) in Definition 5 and $(\mathcal{T}, \mathcal{X})$ is still a tree decomposition for U.*

Proof. Since v is a dangling vertex, all its neighbors are in the bag X_α. So $(\mathcal{T}, \mathcal{X})$ is still a tree decomposition for U. By Observation 14, the resulting graph satisfies the two properties. ◀

We start with a succinct decomposition, which can be obtained without changing U by Observation 16. Recall the two properties of a succinct decomposition:

(i) For every two adjacent nodes i and j in $\mathcal{T}$, the two corresponding bags X_i and X_j satisfy that $X_i \not\subseteq X_j$ and $X_j \not\subseteq X_i$.

(ii) For every vertex v, let $\mathcal{T}_v$ be the subtree of $\mathcal{T}$ consisting of all nodes whose corresponding bags contain v. For every leaf i of $\mathcal{T}_v$, the vertex v must have at least one neighbor in X_i that is not in any other bags of $\mathcal{T}_v$.

Phase 1:

In this phase, we will modify U and $(\mathcal{T}, \mathcal{X})$ so that they satisfy the following properties:

(iii) At least one forget vertex of every leaf node is in S.

(iv) For every node α of degree two in $\mathcal{T}$, let Z_1 and Z_2 be the two legs of torso H_α. Then we have either (a) $X_\alpha = V(Z_1) \cup V(Z_2)$ or (b) there is one vertex in X_α that is also in $S \setminus (V(Z_1) \cup V(Z_2))$.

We first consider property (iii). Let α be a leaf node of $\mathcal{T}$ and β be the parent node of α. By definition of torso, $H_\alpha[X_\alpha \cap X_\beta]$ is a clique. Recall $X_\alpha \setminus X_\beta$ is the set of forget vertices of α. If vertices in $X_\alpha \setminus X_\beta$ are all Steiner vertices, then $X_\alpha \setminus X_\beta$ is an independent set of H_α since virtual edges of H_α are between vertices of $X_\alpha \cap X_\beta$. We apply $\Delta(v)$ for each vertex v in $X_\alpha \setminus X_\beta$. Since each vertex v is a forget vertex of a leaf node and also a dangling vertex, by Lemma 28 the resulting graph satisfies properties (1) and (3) of the exchange graph. We then remove node α from the tree $\mathcal{T}$. This operation preserves h-nearly-embeddability of H_β since $X_\alpha \cap X_\beta$ forms a clique in torso H_β. By Observation 16, we can still maintain the succinctness of $(\mathcal{T}, \mathcal{X})$. We then apply this modification to all leaf nodes in the tree decomposition so that property (iii) is satisfied.

We now consider property (iv). We need to further modify the tree decomposition $(\mathcal{T}, \mathcal{X})$ and the underlying graph U. Our modification might increase the constant h for the decomposition by constant times. Let α be a node that violates property (iv). Then vertices in $Y = X_\alpha \setminus (V(Z_1) \cup V(Z_2))$ are all dangling Steiner vertices. Recall there is no virtual edge between vertices in Y, so Y is an independent set in H_α. For each vertex v in Y, we apply $\Delta(v)$ and consider $V(Z_1) \cup V(Z_2)$ as the apex set of H_α (graphs $G_0, \ldots, G_h$ are all empty in this case). Since each vertex in Y is a dangling vertex, by Lemma 28, the resulting graph satisfies properties (1) and (3) of the exchange graph. We consider two subcases:

1. $V(Z_1) \neq V(Z_2)$. We can maintain succinctness by Observation 16. The apex set of H_α now can have up to $2h$ vertices since each leg has at most h vertices. Thus, H_α is $2h$-nearly-embeddable.
2. $V(Z_1) = V(Z_2)$. If the resulting X_α is a subset of bags of its neighbors in $\mathcal{T}$, then we contract α to its one neighbor. Since Z_1 and Z_2 are cliques in the torso, the contraction does not affect the h-nearly-embeddability of the neighbors of α. After the contraction, the succinctness of $(\mathcal{T}, \mathcal{X})$ is preserved.

We can apply this modification to every node of $\mathcal{T}$ that violates property (iv). After that, the result will become a h_0-RS-tree decomposition for $h_0 = 2h$. We note that all Steiner vertices in the resulting graph U still have degree at least 3 after this Phase, since we only remove Steiner vertices and since there is no edge between two Steiner vertices.

Phase 2:

In this phase, we further remove Steiner vertices from the surface embeddable part of torsos while maintaining the previous properties and the constant h_0 for the decomposition. We only consider such node α that the surface embedded part G_0 of H_α is not empty.

Let $\beta_1, \beta_2, \ldots, \beta_p$ be the neighbors of α in $\mathcal{T}$ where β_p is the parent of α. Let $Z_1, Z_2, \ldots, Z_p$ be legs of H_α where $V(Z_i) = X_\alpha \cap X_{\beta_i}$ for $1 \leq i \leq p$. Let $(A, G_0, G_1, \ldots, G_{h_0})$ be an h_0-nearly-embeddable decomposition of H_α and $C_1, C_2, \ldots, C_{h_0}$ be the corresponding boundary cycles of G_0. Let $Z = Z_1 \cup Z_2 \cup \ldots \cup Z_p$ and $C = C_1 \cup \ldots \cup C_{h_0}$. In this phase, we will make each node in $\mathcal{T}$ satisfying the following property.

(v) If G_0 is not empty, then every Steiner vertex in $V(G_0) \setminus (V(Z) \cup V(C))$ has degree at least 3 in G_0.

For each Steiner vertex v of $V(G_0) \setminus (V(Z) \cup V(C))$ that has degree at most 2 in G_0, we apply $\Delta(v)$. Since each v is a dangling vertex, by Lemma 28 the resulting graph satisfies properties (1) and (3) of exchange graph. Since there are at most two neighbors of v in G_0, the operation $\Delta(v)$ will maintain the genus of G_0 and the resulting torso H_α is still h_0-nearly-embeddable. This modification will maintain properties (iii) and (iv) in Phase 1 since we do not remove any non-Steiner vertex. We can also maintain the succinctness by Observation 16.

We repeatedly apply the above modification to every node of $\mathcal{T}$ and call the final resulting h_0-RS-decomposition $(\mathcal{T}, \mathcal{X})$ *S-succinct*. Now we can bound the size of each bag in the decomposition.

▶ **Lemma 29.** *Let α be a node in the S-succinct tree decomposition $(\mathcal{T}, \mathcal{X})$ and $Z_1, \ldots, Z_p$ be legs of torso H_α. Let $S_\alpha = S \cap H_\alpha$. Then there is a constant c_2 that only depends on $|V(H)|$ such that $V(H_\alpha) \leq c_2|S_\alpha \cup V(Z_1) \cup \ldots \cup V(Z_p)|$.*

Proof. Let $(A, G_0, G_1, \ldots, G_{h_0})$ be an h_0-RS-nearly-embeddable decomposition of H_α and $Z = Z_1 \cup Z_2 \cup \ldots \cup Z_p$. If $G_0 = \emptyset$, then $V(H_\alpha) = A$ has size at most h_0, and the lemma holds trivially for $c_2 = h_0$, since $V(Z)$ and S_α cannot be empty at the same time. Thus, we can assume that G_0 is not empty.

We first bound the number of Steiner vertices of G_0. For each $1 \le i \le h_0$, we add a vertex x_i inside the disk enclosed by the boundary cycle C_i and then add an edge from x_i to each vertex of C_i. Let G'_0 be the resulting graph. Note that if H_α has no vortex, we add no vertex to G_0 and $G'_0 = G_0$ in this case. Let X be the set of added vertices x_i and Y be the set of Steiner vertices of $V(G'_0) \setminus (X \cup V(Z))$. Since virtual edges are between vertices of $V(Z)$, set Y is an independent set of G'_0. By property (v), if a vertex of Y is not in any boundary cycle C_i, then it has degree at least 3 in G'_0. Further, any vertex of $Y \cap C_i$ has degree at least 3 in G'_0 after we adding vertices in X. So every vertex of Y has degree at least 3 in G'_0. By Lemma 4, we can bound the size of Y: $|Y| \le (1 + 4h_0)|V(G'_0) \setminus Y|$. We have:

$$\begin{aligned}
|V(G_0)| &= |Y| + |S_\alpha \cup V(Z)| \\
&\le (1 + 4h_0)(|S_\alpha \cup V(Z)| + |X|) + |S_\alpha \cup V(Z)| && (V(G'_0) \setminus Y = S_\alpha \cup V(Z) \cup X) \\
&\le (4h_0 + 2)|S_\alpha \cup V(Z)| + (1 + 4h_0)h_0 && (|X| \le h_0) \\
&\le (4h_0 + 2 + (1 + 4h_0)h_0)|S_\alpha \cup V(Z)| && (|S_\alpha \cup V(Z)| \ge 1) \\
&= O(h_0^2)|S_\alpha \cup V(Z)|
\end{aligned} \tag{8}$$

Now we bound the size of vortices. By the definition of vortices, for any $1 \le i \le h_0$, there is a path decomposition of width h_0 for G_i where each bag of the path decomposition contains exactly one distinct vertex of $V(C_i)$. Since each bag has at most h_0 vertices, we have $|V(G_i)| \le h_0|V(C_i)|$ for $1 \le i \le h_0$. Since each vertex of C_i is shared by at most h_0 other boundary cycles, we have:

$$|V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{h_0})| \le h_0^2|V(C_1) \cup \ldots \cup V(C_{h_0})| \le h_0^2|V(G_0)|. \tag{9}$$

By combining Equation (8) and Equation (9), we have:

$$\begin{aligned}
|V(H_\alpha)| &\le |V(A)| + |V(G_0)| + |\textstyle\sum_{i=1}^{h_0} V(G_i)| \\
&\le |V(A)| + (h_0^2 + 1)|V(G_0)| && \text{(Equation (9))} \\
&\le |V(A)| + O(h_0^4)|S_\alpha \cup V(Z)| && \text{(Equation (8))} \\
&\le h_0 + O(h_0^4)|S_\alpha \cup V(Z)| \\
&\le \left(h_0 + O(h_0^4)\right)|S_\alpha \cup V(Z)| && (|S_\alpha \cup V(Z)| \ge 1) \\
&= O(h_0^4)(|S_\alpha \cup V(Z)|)
\end{aligned}$$

The lemma follows by setting $c_2 = \Omega(h_0^4)$. ◀

▶ **Lemma 30.** *If U has an S-succinct h_0-RS-decomposition $(\mathcal{T}, \mathcal{X})$, then $|V(U)| \le c_3|S|$ for some constant c_3 that depends on $|V(H)|$ only.*

Proof. The proof follows the similar idea in the proof of Lemma 17. That is, we apply an amortized argument to bound the size of U by the size of S.

We collect vertices of S into a set C during a post-order traversal of the tree $\mathcal{T}$. Initially, $C = \emptyset$. During the collection, we assign each Steiner vertex to the same vertex in C and may mark some leaf nodes of $\mathcal{T}$ as *unavailable*. Initially, all leaf nodes of $\mathcal{T}$ are marked *available*. We will show that there are only a constant number (depending on $|V(H)|$) of distinct Steiner vertices assigned to any vertex in C. During the traversal, we would maintain the following invariant:

Marking invariant: For each non-leaf node α whose parent is not visited, there is at least one available leaf of $\mathcal{T}$ that is a descendant of α.

For a node α, we denote by $\mathcal{T}[\alpha]$ the subtree of $\mathcal{T}$ rooted at α and by $\mathcal{X}[\alpha]$ the union of bags corresponding to nodes in $\mathcal{T}[\alpha]$. Let α be the node of $\mathcal{T}$ that we are currently visiting. We have three cases depending on the number of children of α: zero, one or at least two. If α is a leaf node or has at least two children, we would show that every Steiner vertex in $\mathcal{X}[\alpha]$ is assigned to a vertex in C. However, if α has only one child, then there will be a situation where we will delay the assignment. Let $S_\alpha = X_\alpha \cap S$.

Case 1: node α is a leaf. Since $(\mathcal{T}, \mathcal{X})$ is S-succinct, S_α must contain at least one forget vertex by property (iii). Thus, $|S_\alpha| \geq 1$. Since α is a leaf node, torso H_α has at most one leg, say Z_1. Since $|V(Z_1)| \leq h_0$, by Lemma 29, there is a constant c_2 such that:

$$|X_\alpha| \leq c_2(|S_\alpha \cup V(Z_1)|) \leq c_2(|S_\alpha| + h_0) \leq c_2(h_0 + 1)|S_\alpha|$$

We add all vertices of S_α into C, and uniformly assign Steiner vertices of X_α to the vertices of S_α. By the above inequality, each vertex of S_α is assigned at most $c_2(h_0 + 1)$ Steiner vertices. The marking invariant trivially holds.

Case 2: node α has exactly one child in $\mathcal{T}$. Let β_1 be the child of α and β_2 be the parent of α. Let Z_1 and Z_2 be two legs of torso H_α such that $V(Z_i) = X_\alpha \cap X_{\beta_i}$ for $i = 1, 2$. By property (iv), we have either $X_\alpha = V(Z_1) \cup V(Z_2)$ or there is a vertex of X_α in $S \setminus (V(Z_1) \cup V(Z_2))$. Let L_2 be the set of unassigned Steiner vertices in $\mathcal{X}[\beta_1]$. Then we have two subcases.

Case 2.1: there is a vertex of X_α in $S \setminus (V(Z_1) \cup V(Z_2))$. Let $Y_\alpha = S_\alpha \setminus (V(Z_1) \cup V(Z_2))$. Since $|Z_i| \leq h_0$ for $i = 1, 2$, by Lemma 29, there is a constant c_2 such that:

$$|X_\alpha| \leq c_2(|S_\alpha \cup V(Z_1) \cup V(Z_2)|) \leq c_2(|Y_\alpha| + 2h_0) \leq c_2(2h_0 + 1)|Y_\alpha|$$

We then add all vertices of Y_α into C, and uniformly assign those unassigned Steiner vertices of X_α to the vertices of Y_α. Further, we assign all vertices of L_2 to an arbitrary vertex of Y_α.

Case 2.2: $X_\alpha = V(Z_1) \cup V(Z_2)$. Since each Z_i has size at most h_0, we know $|X_\alpha| \leq 2h_0$. Let S'_α be the set of all vertices in S_α that are currently not in C or are forget vertices of X_α. If S'_α is not empty, we add all vertices of $S'_\alpha \setminus C$ into C and uniformly assign those unassigned Steiner vertices in X_α to vertices of S'_α in C. We also assign vertices of L_2 to an arbitrary vertex in S'_α. If S'_α is empty, we *skip* the node α and do nothing. The marking invariant holds at α inductively since we do not mark any leave of $\mathcal{T}$ in this case.

Case 3: node α has at least two children in $\mathcal{T}$. Let $\beta_1, \beta_2, \ldots, \beta_p$ be the neighbors of α where β_p is the parent of α in $\mathcal{T}$. Recall that by the marking invariant, each subtree $\mathcal{T}[\beta_i]$ has at least one available leaf node. Let γ_i be an available leaf node in the subtree $\mathcal{T}[\beta_i]$ for $1 \leq i \leq p-2$. Let v_i be a forget vertex of γ_i in S for $1 \leq i \leq p-2$. Such vertex v_i exists by property (iii) of the S-succinctness.

Let Z_i be the legs of torso H_α such that $V(Z_i) = X_\alpha \cap X_{\beta_i}$ for $1 \leq i \leq p$ and let $W_\alpha = S_\alpha \setminus (V(Z_1) \cup V(Z_2) \ldots \cup V(Z_p))$. Note that W_α could be empty. By Lemma 29, there is a constant c_2 such that:

$$X_\alpha \leq c_2(|W_\alpha| + \sum_{i=1}^{p} |V(Z_i)|) \leq c_2(|W_\alpha|) + c_2 p h_0 \tag{10}$$

We will add all vertices of W_α into C and then assign those unassigned Steiner vertices of X_α to vertices of W_α so that each vertex of W_α is assigned at most c_2 Steiner vertices. By Equation (10), there are at most cph unassigned Steiner vertices in X_α. We uniformly assign those remaining Steiner vertices to the vertices $v_1, v_2 \ldots, v_{p-2}$. For each vertex v_i, the number of Steiner vertices assigned in this way is at most:

$$\left\lceil \frac{c_2 p h_0}{p-2} \right\rceil \leq 3c_2h_0 + 1 \tag{11}$$

Let L_3 be the set of all unassigned Steiner vertices of $\mathcal{X}[\alpha] \setminus X_\alpha$. We then uniformly assign Steiner vertices of L_3 to vertices $v_1, v_2 \ldots, v_{p-2}$. Finally, we mark node γ_i unavailable for $1 \leq i \leq p-2$. The marking invariant holds since leaf γ_{p-1} remains available. By construction, we have:

▶ Observation 31. If node α is not skipped, then every Steiner vertices in $\mathcal{X}[\alpha]$ is assigned to a vertex in C after we visiting α.

We can bound the number of unassigned Steiner vertices in Case 2.

▶ Claim 32. For Case 2, we have $|L_2| \leq 2h_0^2$ for any node.

Proof. The proof is similar to that of Claim 19. Let α be the current vertex we visit, β_1 be the child of α and Z_1 be the leg of torso H_α so that $V(Z_1) = X_\alpha \cap X_{\beta_1}$. We will map each vertex of L_2 into only one vertex of $S \cap V(Z_1)$ such that there are at most $2h_0$ vertices of L_2 mapped to the same vertex. Then the claim follows from $|V(Z_1)| \leq h_0$.

Let v be a vertex of L_2 and v^* be a node of $\mathcal{T}$ such that v is a forget vertex of v^*. We will map v to a vertex u in $S \cap V(Z_1)$. By Observation 31, all nodes in the subpath between v^* and β_1 of $\mathcal{T}$ are skipped. So they all have degree 2. Since v is a forget vertex of node v^*, node v^* must be a leaf of $\mathcal{T}_v$, the subtree of $\mathcal{T}$ consisting of all nodes whose bags contain v. By condition (ii) of succinctness, vertex v must have a neighbor in S that is an introduced vertex of v^*. Let u be such a neighbor of v. Since v^* is skipped, vertex u must be in the set C when we visit v^*. So there must be another node u^* such that u is an introduced vertex of u^* and we visit u^* before v^*. Since vertex u is an introduced vertex of v^*, we know u^* is not in the subtree $\mathcal{T}[v^*]$. Further, node u^* cannot be in the subtree $\mathcal{T}[\alpha]$ since all nodes between v^* and α have degree 2. Then by the third condition of tree decomposition (Definition 12), vertex u must be in X_α and then $V(Z_1)$. Further, it can be introduced only once in the subtree $\mathcal{T}[\alpha]$.

Since v belongs to L_2, we know v^* is skipped, that is, v^* belongs to Case 2.2. Then the bag X_{v^*} consists of only two legs and $|X_{v^*}| \leq 2h_0$ since any leg of a torso has at most h_0 vertices. So there are at most $2h_0$ vertices mapped to u, a vertex in $S \cap V(Z_1)$. ◀

▶ Claim 33. For Case 3, we have $|L_3| \leq 2(p-1)h_0^2$ for any node α with p neighbors.

Proof. Let β_i be the children of α for $1 \leq i \leq p-1$. By the same argument as that for Claim 32, we can show that there are at most $2h_0^2$ vertices in L_3 that belong to $\mathcal{X}[\beta_i]$ for each $1 \leq i \leq p-1$. This implies the claim. ◀

Now we are ready to bound the number of Steiner vertices assigned to any vertex in C. When a vertex is added into C, the number of Steiner vertices assigned to it is at most $c_2(h_0+1)$ in Case 1, at most $c_2(2h_0+1) + 2h_0^2$ in Case 2 by Claim 32 and at most c_2 in Case 3. After that, we only assign more Steiner vertices to a vertex v in two situations: (a) when we visit a node α in Case 2.2 and vertex v is a forget vertex of α and (b) when we visit a node α in Case 3 and vertex v is a forget vertex of an available leaf node which is a descendant of α. In the former case, we assign at most $2h_0 + 2h_0^2$ Steiner vertices to v where $2h_0$ vertices are from X_α in Case 2.2 and $2h_0^2$ vertices are from L_2 by Claim 32. In latter case, the number of vertices we assign to v is at most:

$$3c_2h_0 + 1 + \left\lceil \frac{2(p-1)h_0^2}{p-2} \right\rceil \leq 3c_2h_0 + 4h_0^2 + 1 \quad (p \geq 3)$$

Thus, each vertex in C is assigned at most $O(c_2h_0)$ Steiner vertices for $c_2 = \Omega(h_0^4)$. The lemma follows by setting $c_3 = \Omega(h_0^5)$. ◀

Now we obtain graph K' from the resulting U by uncontracting each Steiner vertex to its corresponding tree in $F \setminus F_0$. This K' is our final exchange graph.

▶ **Lemma 34.** *If G is an H-minor-free graph, then K' is an r-divisible exchange graph for Feedback Vertex Set problem in G.*

Proof. Let U be the graph obtained after the two-phase modification. By Equation (7) and Lemma 30, we have:

$$|V(U)| \leq c_3|S| \leq c_3c_1(|\mathrm{O} \cup \mathrm{L}|)$$

where c_1 and c_3 are two constants that depend on $|V(H)|$ only. Since each Steiner vertex in U has degree at most $O_H(1)$, its corresponding tree in $F \setminus F_0$ has size $O_H(1)$ by Equation (4). Thus, we have $|V(K')| = O_H(|\mathrm{O} \cup \mathrm{L}|)$, which implies K' is an exchange graph.

Now we construct an r-division of K'. Since U has an h-RS-decomposition where h only depends on $|V(H)|$, graph U is $K_{h'}$-minor-free by Lemma 26, where h' only depends on h and then $|V(H)|$. So by Theorem 2, graph U has an r-division. We start from an r-division of U. If a Steiner vertex is a boundary vertex, we add every vertex in its corresponding tree to the boundary. Otherwise, we add every vertex of the corresponding tree to the region containing the Steiner vertex. Since each tree has $O_H(1)$ size, the result is an r-division of K'. ◀

5 Negative results

We show by examples, that the simple local search with constant exchanges cannot give a PTAS for these two problems in planar graphs. Indeed, our examples show that local search cannot even give a constant approximation. The examples can be constructed from a $k \times k$ grid. See Figure 2.

References

1 V. Bafna, P. Berman, and T. Fujito. A 2-approximation algorithm for the undirected feedback vertex set problem. *SIAM Journal on Discrete Mathematics*, 12(3):289–297, 1999.

2 B. S. Baker. Approximation algorithms for np-complete problems on planar graphs. *Journal of the ACM*, 41(1):153–180, 1994. doi:10.1145/174644.174650.

3 A. Becker and D. Geiger. Optimization of pearl's method of conditioning and greedy-like approximation algorithms for the vertex feedback set problem. *Artificial Intelligence*, 83(1):167–188, 1996.

4 P. Biswal, J. R. Lee, and S. Rao. Eigenvalue bounds, spectral partitioning, and metrical deformations via flows. *Journal of the ACM*, 57(3):13, 2010.

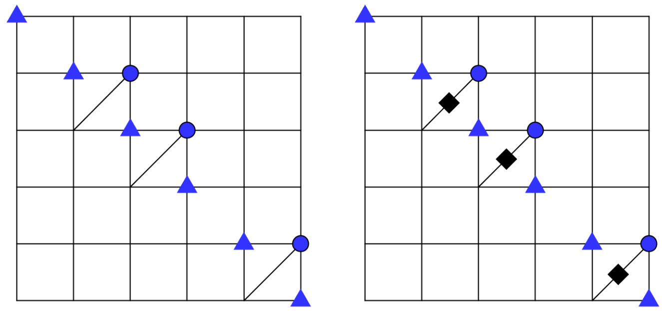

Figure 2 Counterexamples for local search on odd cycle transversal and subset feedback vertex set. Circle vertices represent vertices of the optimal solution, and triangle vertices represent vertices of the local search solution. The grid could be arbitrarily large. We add one edge in some diagonal cells of the grid. Left: counterexample for odd cycle transversal. Since any grid is bipartite and does not contain any odd cycle, any odd cycle in the example must contain an edge in the diagonal cell. All the vertices in the diagonal, represented by triangles, give a solution that is locally optimal, that is, we cannot improve this solution by changing a small number of vertices. This is because each triangle vertex and each new edge, together with some other edges, can form at least one odd cycle in the graph. For a constant k that is smaller than the size of optimal solution, if we remove k triangle vertices in the locally optimal solution, there will be k vertex-disjoint odd cycles in the resulting graph, each of which contains one removed triangle. Then the ratio between the two solutions could be arbitrarily big if the gird is arbitrarily big and the number of new edges is sublinear to the size of the diagonal. Right: counterexample for subset feedback vertex set. The diamonds represent the vertices in the given set U. Similarly, any cycle through a given vertex must contain the two edges in the diagonal cell. By the same reason, the local search solution cannot be improved.

5 H. L. Bodlaender. A linear time algorithm for finding tree-decompositions of small treewidth. In *Proceedings of the 25th Annual ACM Symposium on Theory of Computing*, pages 226–234. ACM, 1993.

6 S. Cabello and D. Gajser. Simple PTAS's for families of graphs excluding a minor. *Discrete Applied Mathematics*, 189(C):41–48, 2015. doi:10.1016/j.dam.2015.03.004.

7 V. Cohen-Addad, É. Colin de Verdière, P. N. Klein, C. Mathieu, and D. Meierfrankenfeld. Approximating connectivity domination in weighted bounded-genus graphs. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, pages 584–597. ACM, 2016.

8 V. Cohen-Addad, P. N. Klein, and C. Mathieu. Local search yields approximation schemes for k-means and k-median in euclidean and minor-free metrics. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*, FOCS' 16, 2016.

9 Daniello. Hadwiger number under matching contraction. Theoretical Computer Science StackExchange (https://cstheory.stackexchange.com/questions/38245/hadwiger-number-under-matching-contraction). Accessed 2017.

10 E. D. Demaine and M. Hajiaghayi. Bidimensionality: New connections between FPT algorithms and PTASs. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA'05, pages 590–601, 2005.

11 H. N. Djidjev. A linear algorithm for par- titioning graphs of fixed genus. *Serdica. Bulgariacae mathematicae publicationes*, 11(4):369–387, 1985.

12 R. G. Downey and M. R. Fellows. *Feasible Mathematics II*, chapter Parameterized Computational Feasibility, pages 219–244. Birkhäuser Boston, 1995. doi:10.1007/978-1-4612-2566-9_7.

13 Guy Even, Joseph Naor, Baruch Schieber, and Leonid Zosin. Approximating minimum subset feedback sets in undirected graphs with applications. *SIAM Journal on Discrete Mathematics*, 13(2):255–267, 2000.

14 F. V. Fomin, D. Lokshtanov, V. Raman, and S. Saurabh. Bidimensionality and EPTAS. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA'11, pages 748–759, 2011.

15 G. Frederickson. Fast algorithms for shortest paths in planar graphs with applications. *SIAM Journal on Computing*, 16:1004–1022, 1987.

16 J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan. A separator theorem for graphs of bounded genus. *Journal of Algorithms*, 5(3):391–407, 1984.

17 S. Har-Peled and K. Quanrud. Approximation algorithms for polynomial-expansion and low-density graphs. In *Algorithms-ESA 2015*, pages 717–728. Springer, 2015.

18 Ken ichi Kawarabayashi and B. A. Reed. A separa- tor theorem in minor-closed classes. In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 153–162, 2010.

19 D. S. Johnson. The NP-completeness column: An ongoing guide (column 19). *Journal of Algorithms*, 8(3):438–448, 1987.

20 R. M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.

21 J. A. Kelner. Spectral partitioning, eigenvalue bounds, and circle packings for graphs of bounded genus. *SIAM Journal on Computing*, 35(4):882–902, 2006.

22 J. Kleinberg and A. Kumar. Wavelength conversion in optical networks. *Journal of Algorithms*, 38:25–50, 2001.

23 R. Lipton and R. Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

24 R. J. Lipton and R. E. Tarjan. Applications of a planar separator theorem. *SIAM Journal on Computing*, 9(3):615–627, 1980. doi:10.1137/0209046.

25 W. Mader. Homomorphiesätze für graphen. *Mathematische Annalen*, 178(2):154–168, 1968. doi:10.1007/BF01350657.

26 T. Nishizeki. On the relationship between the genus and the cardinality of the maximum matchings of a graph. *Discrete Mathematics*, 25(2):149–156, 1979.

27 A. Noga, P. D. Seymour, and T. Robin. A separator theorem for nonplanar graphs. *Journal of the American Mathematical Society*, 3(4):801–808, 1990.

28 B. A. Reed and D. R. Wood. A linear-time algorithm to find a separator in a graph excluding a minor. *ACM Transactions on Algorithms*, 5(4):39, 2009.

29 G. Ringel and J. WT Youngs. Solution of the Heawood map-coloring problem. *Proceedings of the National Academy of Sciences*, 60(2):438–445, 1968.

30 N. Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *Journal of Combbinatoral Theory Series B*, 89(1):43–76, 2003. doi:10.1016/S0095-8956(03)00042-X.

31 C. Wulff-Nilsen. Separator theorems for minor-free and shallow minor-free graphs with applications. In *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*, pages 37–46, 2011.

32 M. Yannakakis. Node-and edge-deletion NP-complete problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 253–264. ACM, 1978.