Reliable Spanners: Locality-Sensitive Orderings Strike Back

Hung Le
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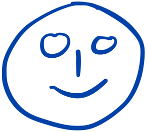

Arnold Filtser
Columbia University

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Disclaimer

- (Lots of?) Buzzwords in this talk:
  - Reliable Spanners, Oblivous Reliable Spanners, Hop-bounded Reliable Left Spanner
  - Locality Sensitive Orderings, Left-Sided Locality-Sensitive Orderings, Triangle Localitive Sensitive Orderings.
  - Sparse Covers, Ultrametric Covers.
  - Doubling Metrics, Planar Metrics, Minor-free Metrics.

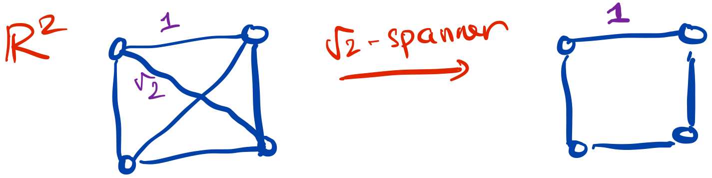
- I try to give the context of these buzzwords and how they are related, glossing over technical details.

Spanners

A t-spanner of a finite metric space (X, d_X), is a weighted graph $H(X, E, w)$ over X where for every pair of points $x, y \in X$:

$$d_X(x, y) \leq d_H(x, y) \leq t \cdot d_X(x, y) \quad (1)$$

d_H is the shortest path metric in H.

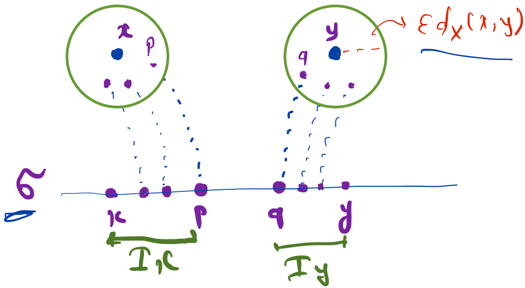


- Parameter t is called the stretch of the spanner.

Locality Sensitive Ordering (LSO) → Chan, Har-Peled, Jones ITCS 19

(τ, ϵ)-**LSO:** Given a metric space (X, d_X), a collection Σ of linear orderings is a (τ, ϵ)-LSO if:

- $|\Sigma| \leq \tau$.
- $\forall x \neq y \in X$, there is an ordering $\sigma \in \Sigma$ such that (w.l.o.g.) $x \prec_\sigma y$ and the points between x and y in σ could be partitioned into two consecutive intervals I_x, I_y where $I_x \subseteq B_X(x, \epsilon \cdot d_X(x,y))$ and $I_y \subseteq B_X(y, \epsilon \cdot d_X(x,y))$.



LSO for $\mathbb{R}^d$

think ε as a constant < 1

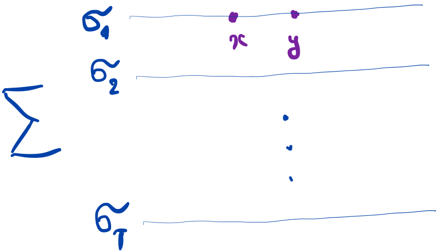
Theorem (CHJ09): Any n-point set in $\mathbb{R}^d$ has a $O(\epsilon^{-d}, \epsilon)$-LSO.

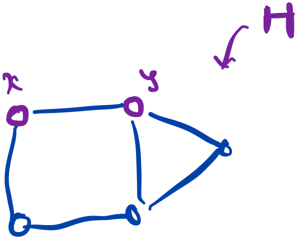
- Chan, Har-Peled, and Jones (CHJ09) provided many applications of LSO in solving: dynamic bichromatic closest pairs, dynamic spanners, dynamic approximate MST, static and dynamic fault-tolerant spanners.
- For doubling metrics, Chan, Har-Peled, and Jones (CHJ09) construct a $\left((\epsilon^{-1}\log n)^{O(d)}, \epsilon\right)$-LSO.

Spanners from LSO

[CHJ19]

Lemma: If (X, d_X) admits a (τ, ϵ)-LSO, it has a $(1 + O(\epsilon))$-spanner with $O(\tau n)$ edges.

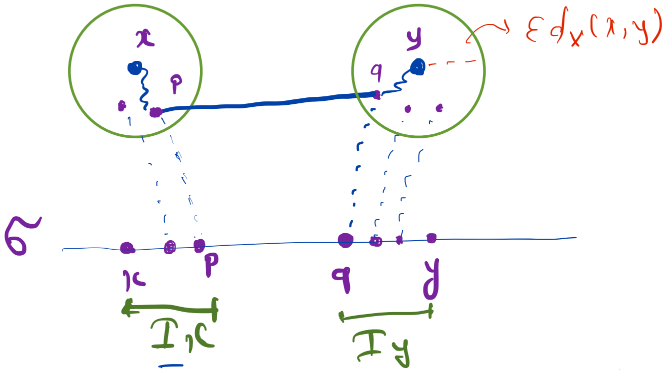




$$w_H(x,y) = d_X(x,y)$$

$$|E(H)| = \tau \cdot (n-1)$$
$$= O(\tau n)$$

Spanners from LSO: The stretch



$\leq \varepsilon d_X(x,y) + \ldots$

$$d_H(x,y) \leq \overbrace{(1+2\epsilon)d_X(x,y)}^{\geq d_X(p,q)} + (d_H(x,p) + d_H(y,q))$$
$$= (1+O(\epsilon))d_X(x,y)$$

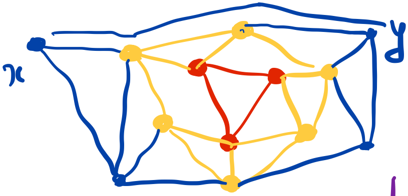
$\leq \varepsilon^2 d_X(x,y) + \varepsilon^3 d_X(x,y) \ldots$

- Stretch guarantee crucially relies on that $\epsilon < 1$.

Deterministic Reliable Spanner

A weighted graph $H(X, E, w)$ is a (deterministic) ϑ-reliable t-spanner of a metric space (X, d_X) if for every set $B \subseteq X$ of points, there is a set $B^+ \supseteq B$, called an *extension* of B, such that:

1. $|B^+| \le (1 + \vartheta)|B|$.
2. For every $x, y \notin B^+$, $d_X(x, y) \le d_{H[X \setminus B]}(x, y) \le t \cdot d_X(x, y)$.



B = {red vertices}

B^+ = {red + yellow vertices}

$|B^+| \le (1+\vartheta)|B|$

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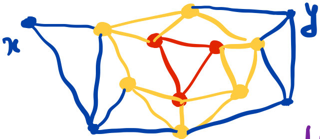
- ϑ is a (small) constant. The size of B could be $\Omega(n)$, and we don't need to know $|B|$ in advance.
  - f-fault tolerant spanners: need to know f in advance, and have $\Omega(n^2)$ edges when $f = \Omega(n)$.



$B = \{\text{red vertices}\}$

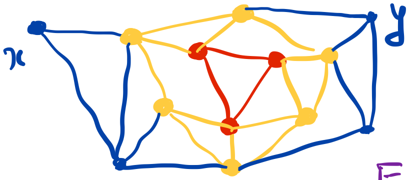
$B^+ = \{\text{red + yellow vertices}\}$

$|B^+| \leq (1+\vartheta)\,|B|$

Oblivious Reliable Spanner

An oblivious ϑ-reliable t-spanner is a distribution $\mathcal{D}$ over t-spanners $H(X, E, w)$ of (X, d_X), such that for every set $B \subseteq X$, there exists B^+ of B such that:

1. $\mathbb{E}_{H\sim\mathcal{D}}\left[|B^+|\right] \leq (1+\vartheta)|B|$
2. For every $x, y \notin B^+$, $d_X(x, y) \leq d_{H[X\setminus B]}(x, y) \leq t \cdot d_X(x, y)$.



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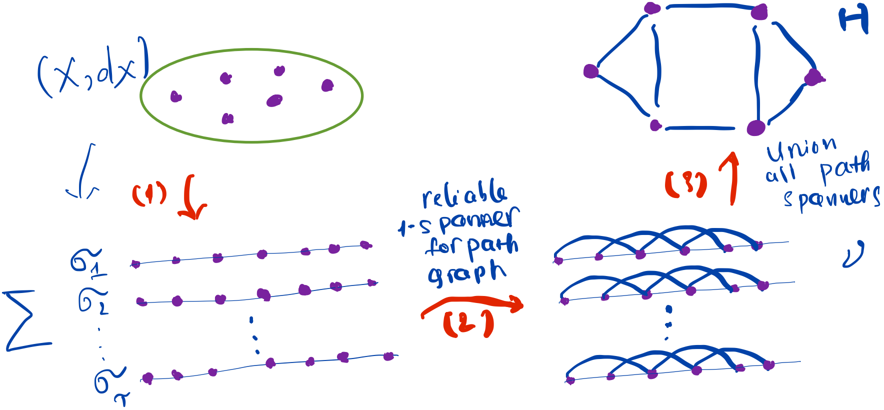
2. For every $x, y \notin B^+$, $d_X(x, y) \leq d_{H[X\setminus B]}(x, y) \leq t \cdot d_X(x, y)$.

- There exists $P \in \mathbb{R}^1$ such that any deterministic reliable t-spanner must have $\Omega(n \log n)$ edges, for any constant t, while it is possible to construct an oblivious ϑ-reliable $(1+\epsilon)$-spanner with $O(n)$ edges.

Reliable Spanners from LSO

If (X, d_X) admits a (τ, ϵ)-LSO, it has a:

- deterministic ϑ-reliable $(1 + O(\epsilon))$-spanner with $\tilde{O}(n\tau^7 \log n)$ edges (for constant ϑ, ϵ) [BHO19].
- oblivious ϑ-reliable $(1 + O(\epsilon))$-spanner with $O(n(\log \log n)\tau^2)$ edges (for constant ϑ, ϵ) [BHO20].



Other Metrics Spaces?

ϑ
ε are small constants

- Doubling metrics of dimension d: ϑ-reliable $(1+\epsilon)$-spanners with $O(n(\log n)^{O(d)})$ edges [BHO19].
- Tree metrics: oblivious ϑ-reliable $(3+\epsilon)$-spanners with $O(n\log n(\log\Delta)^2)$ edges [HO20].
- Planar metrics: oblivious ϑ-reliable $(3+\epsilon)$-spanners with $O(n\log^2 n(\log\Delta)^2)$ edges [HO20].
- General metrics: oblivious ϑ-reliable $O(k)$-spanners with $O(n^{1+1/k}\log n(\log\Delta)^2)$ edges [HO20].

Sparse Cover based Construction

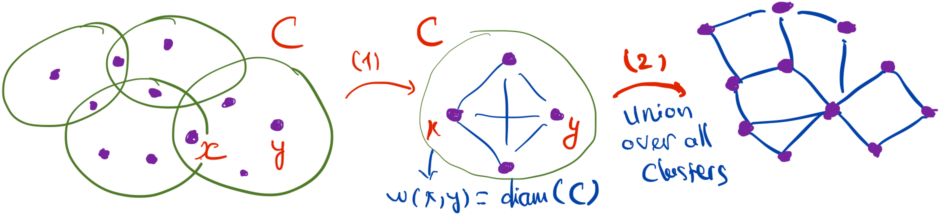
Δ is the spread of the metric: $\Delta = \frac{\max_{x,y} d_X(x,y)}{\min_{x\neq y} d_X(x,y)}$.

- Δ could be $2^{\Omega(n)}$, or worst, and the dependency on $\log(\Delta)$ is inherent in the technique of Har-Peled and Oláh [HO20].

Sparse Cover

A (τ, ρ)-cover is a collection $\mathcal{C}$ of clusters such that:

- every point belongs to at most τ clusters.
- for every pair $x, y \in X$ there is a cluster $C \in \mathcal{C}$ such that $x, y \in C$ and $\frac{1}{\rho} \cdot \mathrm{diam}(C) \le d_X(x,y) \le \mathrm{diam}(C)$.



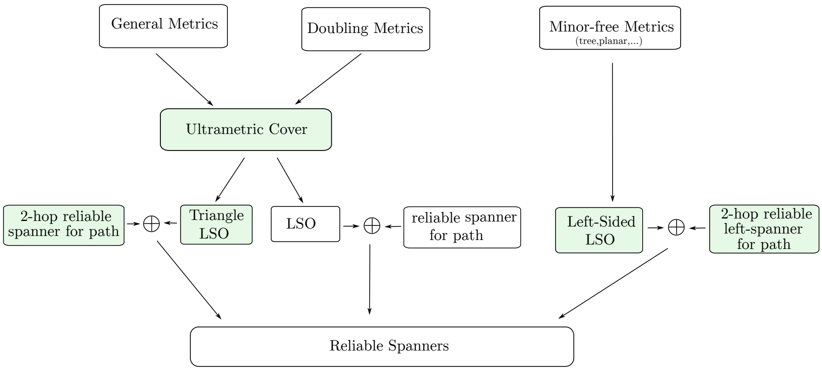
Theorem: (τ, ρ)-cover implies oblivious ϑ-reliable 2ρ-spanner with $O(n\tau^2 \log \tau)$ edges.

- Tree metrics and planar metrics have $(\log(\Delta)\epsilon^{-1}, 2+\epsilon)$-cover.
- General metrics have $(\log(\Delta)n^{1/k}, O(k))$-cover.

Our Results

Family	stretch	guarantee	size	ref
Doubling dim. d	$1+\epsilon$	Determi.	$n\tilde{O}(\log n)$	This paper
	$1+\epsilon$	Oblivious	$n\tilde{O}(\log\log n)^2$	This paper
General metric	$512\cdot k$	Oblivious	$O(n^{1+\frac{1}{k}}k\log(n)\log^2\Delta)$	[HO20]
	$8k-2$	Oblivious	$O(n^{1+\frac{1}{k}}k^2\tilde{O}(\log n)$	This paper
Tree	$3+\epsilon$	Oblivious	$n\cdot\tilde{O}(\log^2 n\log^2\Delta)$	[HO20]
	2	Oblivious	$n\cdot O(\log^3 n)$	This paper
	$t<2$	Oblivious	$\Omega(n^2)$	This paper
Planar	$3+\epsilon$	Oblivious	$n\cdot\tilde{O}(\log^2 n\log^2\Delta)$	[HO20]
	$2+\epsilon$	Oblivious	$n\cdot O(\log^5 n)$	This paper
Minor-free	$2+\epsilon$	Oblivious	$n\cdot O(\log^5 n)$	This paper

Our Techniques: LSO (strikes back)

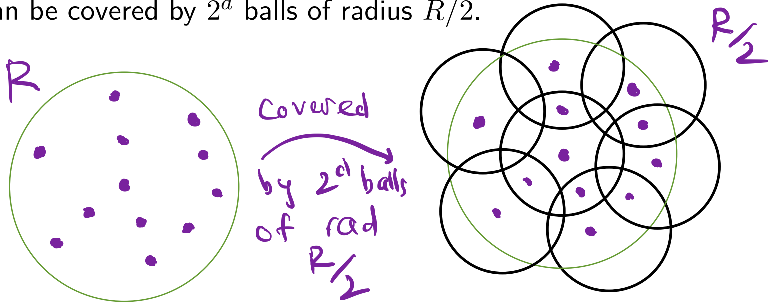


The Rest of the Talk

- Reliable Spanners for Doubling Metrics.
- Reliable Spanners for Tree Metrics.

Doubling Metrics

A metric (X, d_X) has doubling dimension d if any ball of radius R can be covered by 2^d balls of radius $R/2$.



- n-point set in $(\mathbb{R}^d, ||\cdot||_2)$ has doubling dimension $\Theta(d)$.

Recall: Reliable Spanners from LSO

If (X, d_X) admits a (τ, ϵ)-LSO, it has a:

- deterministic ϑ-reliable $(1 + O(\epsilon))$-spanner with $\tilde{O}(n\tau^7 \log n)$ edges (for constant ϑ, ϵ) [BHO19].
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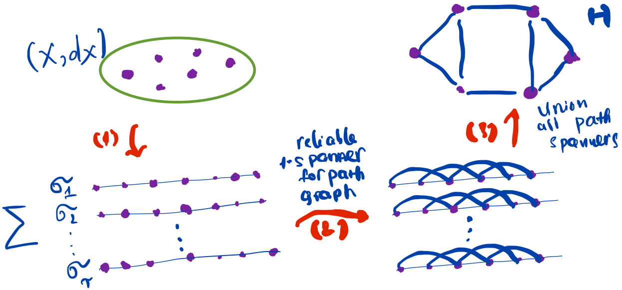


Doubling Metrics Admit LSO

Theorem (FL20): For every $\epsilon \in (0, \frac{1}{6})$, every metric space (X, d_X) of doubling dimension d admits an $\left(\epsilon^{-O(d)}, \epsilon\right)$-LSO.

Our technique: Ultrametric Cover!

- Previous result [CHJ19]: (X, d_X) of doubling dimension d admits an $\left((\log n\epsilon^{-1})^d, \epsilon\right)$-LSO.

Corollary: If (X, d_X) has doubling dimension d, it has

- deterministic ϑ-reliable $(1 + O(\epsilon))$-spanner with $\tilde{O}(n \log n)$ edges (for constant ϑ, ϵ)
- oblivious ϑ-reliable $(1 + O(\epsilon))$-spanner with $O(n(\log \log n)^2)$ edges (for constant ϑ, ϵ)

Ultrametrics

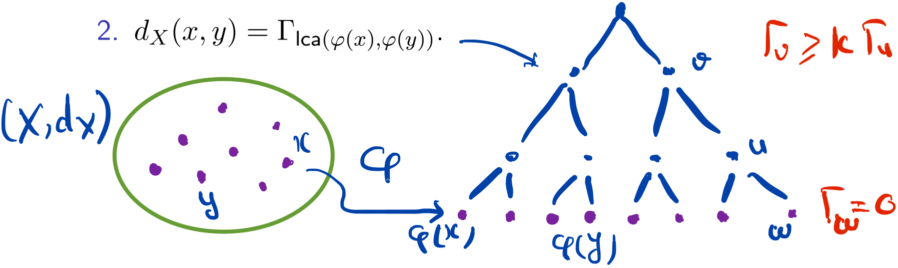
- An ultrametric is a metric with a strengthened triangle inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \forall x,y,z \in X$$

- An ultrametric can be repsented by a k-HST:

k-**HST**: A metric (X, d_X) is a k-hierarchical well-separated tree if there exists a bijection φ from X to leaves of a rooted tree T:

1. Each node $v \in T$ has a label Γ_v s,t $\Gamma_v = 0$ if v is a leaf and $\Gamma_v \geq k\Gamma_u$ if u is any child of v.
2. $d_X(x,y) = \Gamma_{\mathsf{lca}(\varphi(x),\varphi(y))}$.



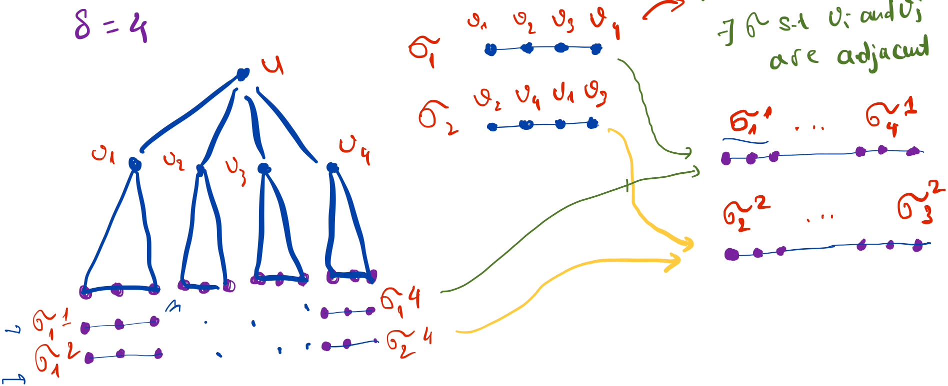
LSO for Ultrametrics

- A (δ, k)-ultrametric is a k-HST where every vertex of the tree T has degree at most δ.

Lemma (FL20): Every (δ, k)-ultrametric (U, d_U) of degree δ admits a $(\left\lceil \frac{\delta}{2} \right\rceil, \frac{1}{k})$-LSO.

[Walecki]

$\forall\, v_i, v_j$
$\exists\, \sigma$ s.t. v_i and v_j are adjacent

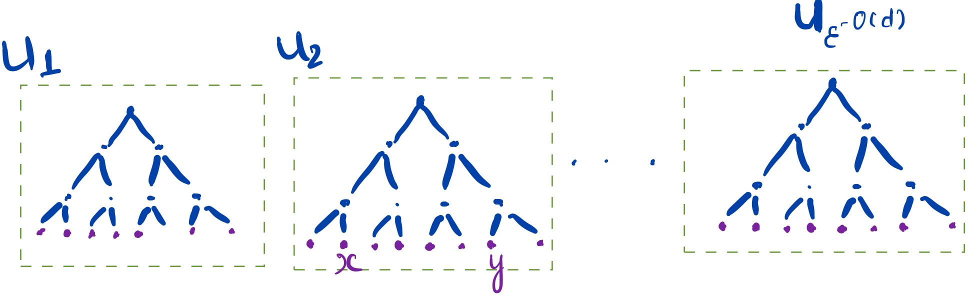


Ultrametric Cover

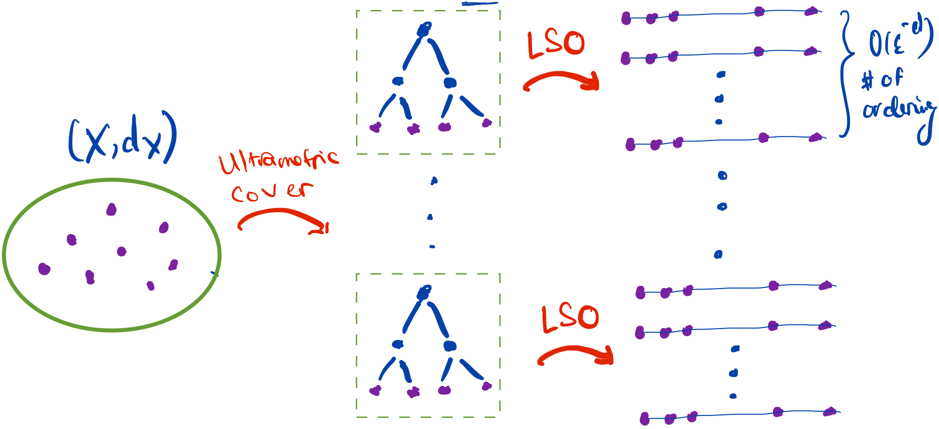
Theorem (FL20): For any metric (X, d_X) of doubling dimension d, we can construct a set $\mathcal{U} = \{U_1, U_2, \ldots\}$ such that:

- $|U| = \epsilon^{-O(d)}$.
- Each U_i is an $(\epsilon^{-O(d)}, \frac{1}{\epsilon})$-ultrametrics.
- For every $x \neq y \in X$, there exists $U_i \in \mathcal{U}$ such that:

$$d_X(x, y) \leq d_{U_i}(x, y) \leq (1 + \epsilon) d_X(x, y)$$



Putting Things Together



Theorem (FL20): For every $\epsilon \in (0, \frac{1}{6})$, every metric space (X, d_X) of doubling dimension d admits an $\left(\epsilon^{-O(d)}, \epsilon\right)$-LSO.

The Rest of the Talk

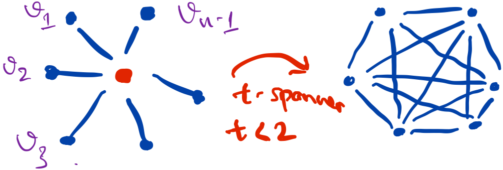
- Reliable Spanners for Doubling Metrics. ✓
- Reliable Spanners for Tree Metrics.

Tree Metrics

- A tree metric (X, d_X) is the shortest path metric of a tree.

Lemma (FL20): Any (oblivious/determnistic) ϑ-reliable t-spanner for $t < 2$ of a star must have $\Omega(n^2)$ edges.

$B = \{ \text{center vertex} \}$



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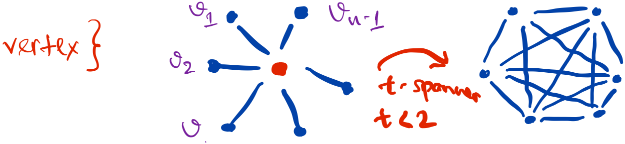


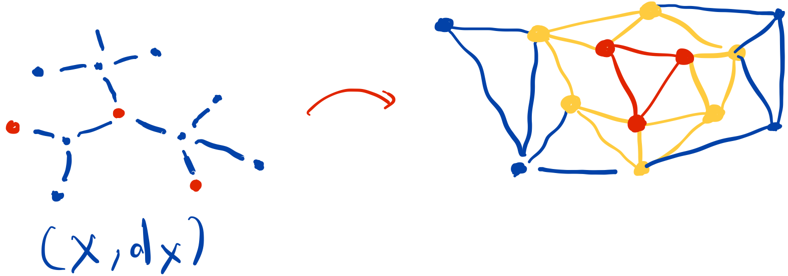
$(2t)$-spanner for $t \geq 1$ of a star must have $\Omega(n^{1+\frac{1}{t}})$ edges [HO20].

- There exists an oblivious ϑ-reliable $(3+\epsilon)$-spanner with $O(n \log^2 n \log^2(\Delta))$ edges.

Question: Can we construct an oblivious ϑ-reliable 2-spanner with $O(n\text{polylog}(n))$ edges?

Tree Metrics: Our Results

Theorem (FL20): Tree metrics have an oblivious ϑ-reliable 2-spanner with $O(n \log^3 n)$ edges.



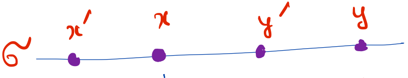
Lemma (FL20): Any (oblivious/determnistic) ϑ-reliable t-spanner for $t < 2$ of a star must have $\Omega(n^2)$ edges.

New Tools (1): Left-sided Locality-Sensitive Ordering

(τ, ρ)**-left-sided LSO** A collection Σ of *partial orderings* of (X, d_X) is a (τ, ρ)-left-sided LSO if:

ordering on subsets of X

- every point $x \in X$ belongs to at most τ orderings in Σ.
- $\forall x, y \in X$, there is an order $\sigma \in \Sigma$ such that for every $x' \preceq_\sigma x$ and $y' \preceq_\sigma y$ it holds that $d_X(x', y') \leq \rho \cdot d_X(x, y)$.



$$d_X(x', y') \leq \rho\, d_X(x, y)$$

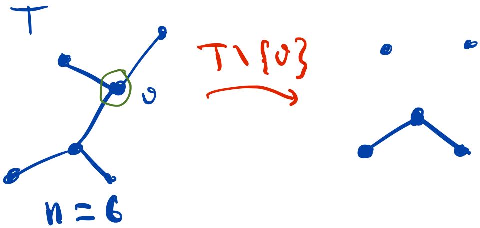
$$\forall\ x' \preceq_\sigma x\ ;\ y' \preceq_\sigma y$$

Left-Sided LSO for Tree

Lemma (FL20): n-point tree metric (X, d_X) has a $(\log n, 1)$-Left-Sided LSO.

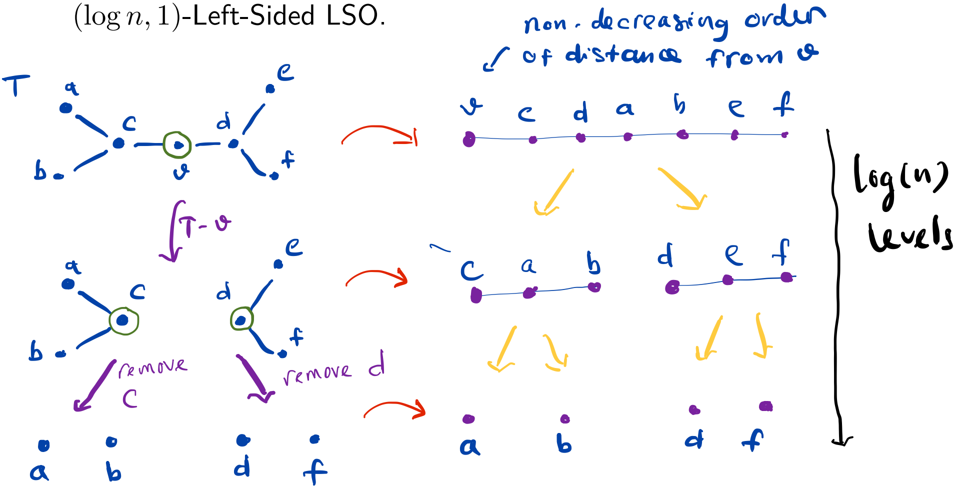
Construction:

- v is a center of a tree T if every connected component of $T \setminus \{v\}$ has at most $n/2$ vertices.



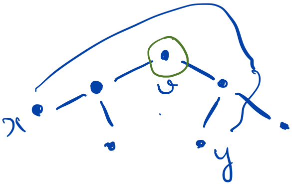
- There always exists a center vertex for a given tree T [Jordan,1869].

Lemma (FL20): n-point tree metric (X, d_X) has a $(\log n, 1)$-Left-Sided LSO.

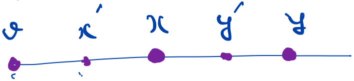


∀ v ∈ X belongs to at most log n orders

$\forall x, y \in X$, there is an order $\sigma \in \Sigma$ such that for every $x' \preceq_\sigma x$ and $y' \preceq_\sigma y$ it holds that $d_X(x', y') \leq \rho \cdot d_X(x, y)$.



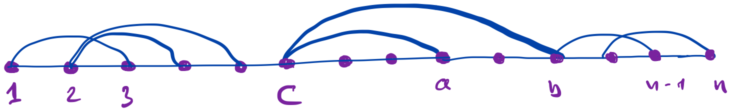
non-decreasing order of distance from v



$$
\begin{aligned}
d_X(x', y') &\leq d_X(v, x') + d_X(v, y') \\
&\leq d_X(v, x) + d_X(v, y) \\
&= d_X(x, y)
\end{aligned}
$$

New Tools (2): 2-Hop Reliable Left Spanner

Given a path graph P_n, a 2-hop left spanner H is a graph such that for every $a < b$ there is $c \leq a, b$, such that $\{c, a\}, \{c, b\} \in H$.



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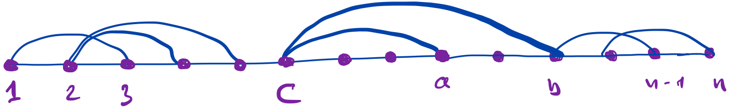
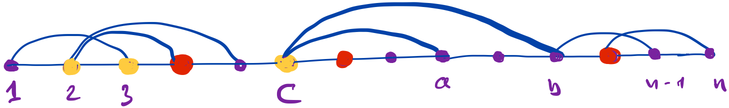


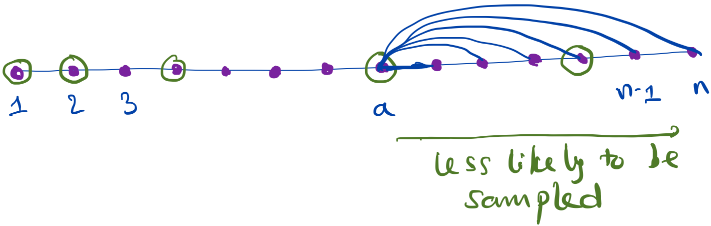
A distribution $\mathcal{D}$ over 2-hop left spanners H is ϑ-reliable if for every attack $B \subseteq X$, there is a set $B \subseteq B^+$ such that $\mathbb{E}[B^+] \leq (1+\vartheta)|B|$, and for every $a < b$ such that $a, b \notin B^+$, there is $c \leq a, b$, $c \notin B$ such that $\{c, a\}, \{c, b\} \in E(H)$.



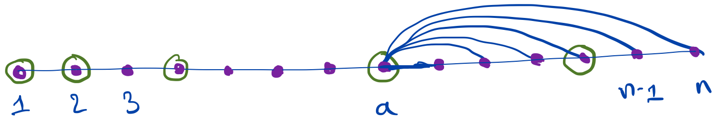
$B = \{$red vertices$\}$, $B^+ = \{$red + yellow vertices$\}$

2-Hop Reliable Left Spanners for Path Graphs

Lemma (FL20): For every $\vartheta \in (0,1)$, the n-vertex path P_n admits an oblivious ϑ-reliable 2-hop left spanner with $n \cdot O(\vartheta^{-1}\log n)$ edges.



- Sample $a \in [n]$ with probability $\frac{c}{a\vartheta}$.
  - N be the set of sampled vertices.
  - If a is sampled, add edge from a to every vertex in $\{a+1, a+2, \ldots, n\}$.

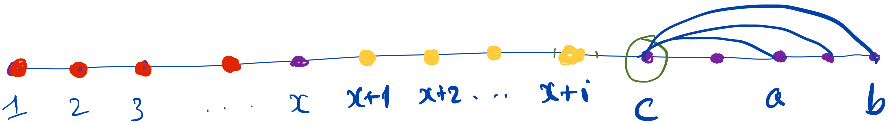


sample with prob $\frac{c}{a\nu}$

less likely to be sampled

$$\#\text{edges} = \sum_{a=1}^{n} \frac{c}{a\nu}(n-a) \le \frac{cn}{\nu}\sum_{i=1}^{n}\frac{1}{a}$$

$$= O\left(\frac{n \lg n}{\nu}\right)$$



- Suppose $B = \{1, 2, \cdots x\}$; $B \subseteq B^+$

$x+i \in B^+$ if $N \cap [x+1, x+i] = \phi$

↳ sampled vertices

- $\forall u, v \in [n] \setminus B^+ : \exists\, c \leq a, b$ s.t. $\{a, c\}, \{b, c\} \in E(H)$

- $$E[|B^+ \setminus B|] = \sum_{i=1}^{n-x} \underbrace{Pr[x+i \in B^+]}_{= \prod_{j=1}^{i} \left(1 - \frac{c}{(x+i)\cdot 2\nu}\right)} = \left(1 + \frac{i}{x}\right)^{-2/\nu}$$

$$\Rightarrow \mathbb{E}[|B^+ \setminus B|] = \sum_{i=1}^{n-x} \left(1+\frac{i}{x}\right)^{-2/\nu}$$

$$= \sum_{s \geq 0} \sum_{i \in [s\nu x, (s+1)\nu x)} \left(1+\frac{i}{x}\right)^{-2/\nu}$$

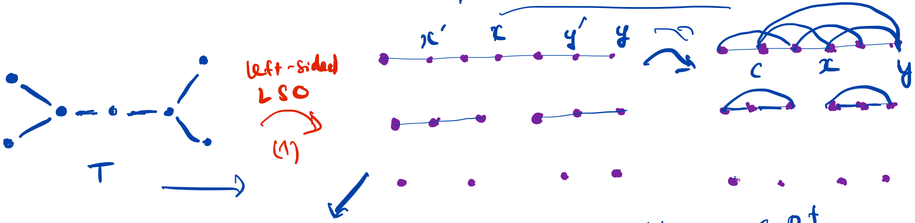
$$= \sum_{s \geq 0} \nu x e^{-s} = O(\nu x)$$

$$= O(\nu)|B|$$

$$\Rightarrow \mathbb{E}[|B^+|] = (1+O(\nu))|B|$$

Putting Things Together

$d_X(x',y') \le d_X(x,y)$



Σ = partial orderings

$\forall v \in$ at most $\log n$ ordering

$\forall\, x, y \in B^+$

$\exists\, c$ and σ st

$\{c, x\}, \{c, y\} \in H$

and $d_X(x,c) \le d_X(x,y)$

$d_X(y,c) \le d_X(x,y)$

$$d_H(x,y) \le d_X(x,c) + d_X(y,c) \le 2d_X(x,y)$$

Theorem (FL20): Tree metrics have an oblivious ϑ-reliable 2-spanner with $O(n \log^3 n)$ edges.

Conclusion

- Reliable Spanners from Locality-Sensitive Orderings.
- Open Problems:
  - Can we construct an oblivious reliable 2-spanner for trees with $O(n)$ edges?
  - Can we construct (deterministic or oblivious) reliable spanners for graphs where the spanner must be the subgraph of the input graph?
  - Several other open problems in our paper, check it out.

Thank you !!!