Light Euclidean Spanners with Steiner Points

Hung Le

University of Massachusetts
Amherst

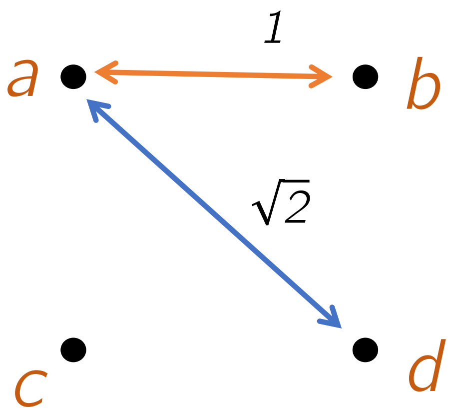
Shay Solomon

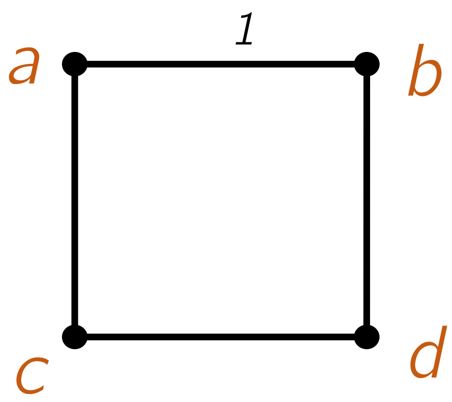
Tel Aviv University

Euclidean Spanners

A Euclidean **t-spanner** of a point set $P \in R^d$ is a **geometric graph** $G(V,E,w)$:

1. $V = P$
2. $\forall e = (p, q), w(e) = ||p, q||_2$
3. $\max_{p \neq q \in V} \frac{d_G(p, q)}{||p, q||_2} \leq t$ t is called **stretch factor**





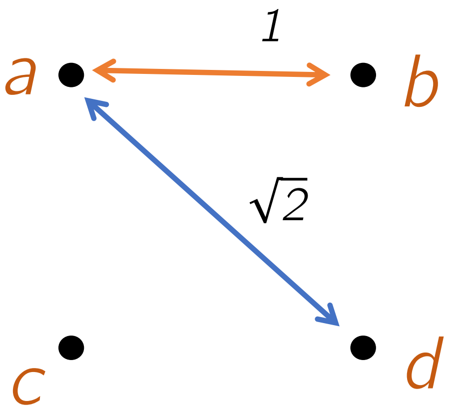
$G(V,E,w)$

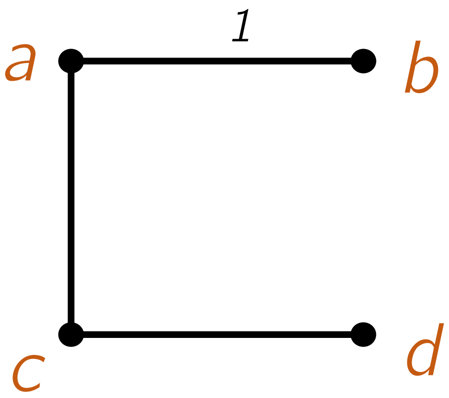
$\sqrt{2}$ – spanner

Sparsity and Lightness

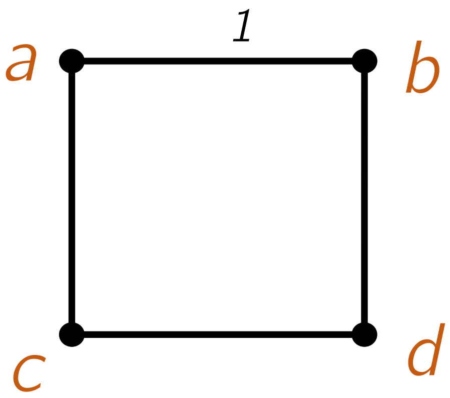
Think of t–spanner as a **sparse and light weight** graph that preserves Euclidean distances up to a factor of t.

$$\text{Sparsity}(G) = \frac{|E(G)|}{|E(\text{MST})|} \qquad \text{Lightness}(G) = \frac{w(G)}{w(\text{MST})}$$





MST



$\sqrt{2}$ – spanner

Sparsity $= 4/3$

Lightness $= 4/3$

Applications of Euclidean spanners

Approximating geometrical graphs via "spanners" and "banyans"

Satish B. Rao & Warren D. Smith
NECI, 4 Independence way, Princeton NJ 08544
{satish,wds}@research.NJ.NEC.COM

Approximation algorithm

Approximate Distance Oracles for Geometric Graphs

Joachim Gudmundsson* Christos Levcopoulos† Giri Narasimhan‡ Michiel Smid§

Distance oracle

EXPLORING PROTEIN FOLDING TRAJECTORIES USING GEOMETRIC SPANNERS

D. RUSSEL and L. GUIBAS
Computer Science Department
353 Serra Mall
Stanford, CA 94305, USA

Computational biology

Sparse Communication Networks and Efficient Routing in the Plane

[Extended Abstract]

Yehuda Hassin
Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot, 76100 Israel.
hassin@wisdom.weizmann.ac.il

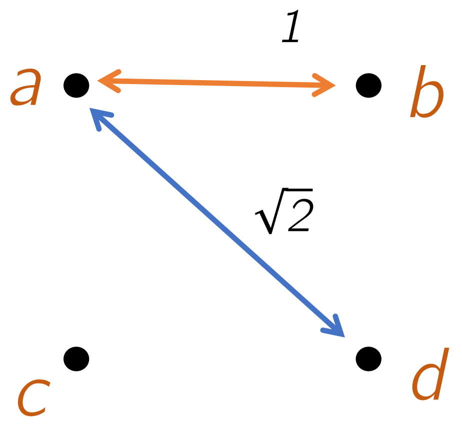
David Peleg*
Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science, Rehovot, 76100 Israel.
peleg@wisdom.weizmann.ac.il

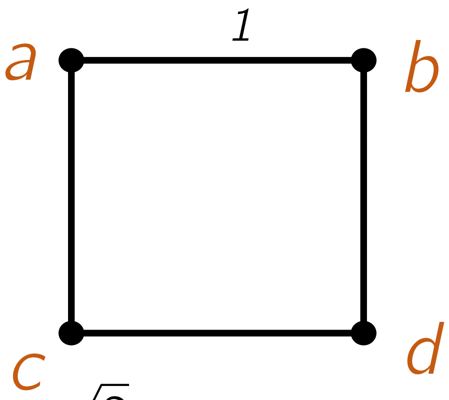
Network design

Steiner Euclidean Spanners

A **Steiner** Euclidean ***t*-spanner** of a point set $P \in \mathbb{R}^2$ is a **geometric graph** $G(V,E,w)$:

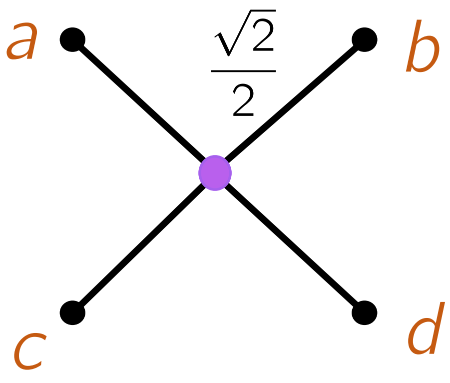
1. $P \subseteq V, Q = V \setminus P$ **Steiner points**
2. $\forall e = (p, q), w(e) = ||p, q||_2$
3. $\max_{p \neq q \in P} \frac{d_G(p, q)}{||p, q||_2} \leq t$





$\sqrt{2}$ – spanner

$w(H) = 4$



$\sqrt{2}$ – spanner

$w(G) = 2\sqrt{2} = 2.8$

Non-Steiner vs Steiner Spanners

Question: Do Steiner points really help?
(in reducing sparsity and lightness)

Spoiler alert: **Yes!!!**

Known Results: R^2

$t = 1 + \varepsilon$

$\tilde{\Omega}, \tilde{O}$ hide *polylog(1/ε)*.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(1/\varepsilon)$ [Clarkson87;Keil88]	$\Omega(1/\varepsilon)$ [LS,FOCS19]	$\tilde{O}(1/\sqrt{\varepsilon})$ [LS,FOCS19]	$\tilde{\Omega}(1/\sqrt{\varepsilon})$ [LS,FOCS19]
Lightness	$\tilde{O}(1/\varepsilon^2)$ [LS,FOCS19]	$\Omega(1/\varepsilon^2)$ [LS,FOCS19]	?	$\tilde{\Omega}(1/\varepsilon)$ [LS,FOCS19]

[LS] H. Le and S. Solomon, "*Truly optimal Euclidean spanners,*" FOCS'19.

This Work

$t = 1 + \varepsilon$

$\tilde{\Omega}, \tilde{O}$ hide $polylog(1/\varepsilon)$.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(1/\varepsilon)$ [Clarkson87;Keil88]	$\Omega(1/\varepsilon)$ [LS,FOCS19]	$\tilde{O}(1/\sqrt{\varepsilon})$ [LS,FOCS19]	$\tilde{\Omega}(1/\sqrt{\varepsilon})$ [LS,FOCS19]
Lightness	$\tilde{O}(1/\varepsilon^2)$ [LS,FOCS19]	$\Omega(1/\varepsilon^2)$ [LS,FOCS19]	$\tilde{O}(1/\varepsilon)$ [LS,ESA20]	$\tilde{\Omega}(1/\varepsilon)$ [LS,FOCS19]

when spread Δ is $poly(1/\varepsilon)$

hold when Δ is $poly(1/\varepsilon)$.

$$\text{Spread } \Delta = \frac{\max_{p,q \in P} ||p,q||}{\min_{p \neq q \in P} ||p,q||}$$

[LS] H. Le and S. Solomon, "*Light Euclidean Spanners with Steiner Points,*" *ESA*'20.

Known Results: R^d when $d \geq 3$

$t = 1 + \varepsilon$

$\tilde{\Omega}$, $\tilde{O}$ hide $polylog(1/\varepsilon)$.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(\varepsilon^{1-d})$ [RS91]	$\Omega(\varepsilon^{1-d})$ [LS,FOCS19]	$\tilde{O}(\varepsilon^{(1-d)/2})$ [LS,FOCS19]	? Lower bound
Lightness	$\tilde{O}(\varepsilon^{-d})$ [LS,FOCS19]	$\Omega(\varepsilon^{-d})$ [LS,FOCS19]	?	? Lower bound

[LS] H. Le and S. Solomon, "*Truly optimal Euclidean spanners,*" FOCS'19.

This Work

$t = 1 + \varepsilon$

$\tilde{\Omega}, \tilde{O}$ hide *polylog(1/ε)*.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(\varepsilon^{1-d})$ [RS91]	$\Omega(\varepsilon^{1-d})$ [LS,FOCS19]	$\tilde{O}(\varepsilon^{(1-d)/2})$ [LS,FOCS19]	? Lower bound
Lightness	$\tilde{O}(\varepsilon^{-d})$ [LS,FOCS19]	$\Omega(\varepsilon^{-d})$ [LS,FOCS19]	$\tilde{O}(\varepsilon^{(-d-1)/2})$ [LS,ESA20]	? Lower bound

when spread Δ is *poly(1/ε)*

hold when Δ is *poly(1/ε)*.

[LS] H. Le and S. Solomon, "*Light Euclidean Spanners with Steiner Points,*" *ESA*'20.

This Work

$t = 1+ \varepsilon$

$\widetilde{\Omega}, \widetilde{O}$ hide *polylog(1/ε)*.

	Non-Steiner Spanners		Steiner Spanners	
	[LS,FOCS19]	[LS,FOCS19]	[LS,ESA20]	[LS,FOCS19]

Steiner points help reducing lightness quadratically

(for point sets with spread poly($1/\varepsilon$))

when spread Δ is *poly(1/ε)*

hold when Δ is *poly(1/ε)*.

$$\text{Spread } \Delta = \frac{\max_{p,q\in P} ||p,q||}{\min_{p\neq q\in P} ||p,q||}$$

[LS] H. Le and S. Solomon, "*Light Euclidean Spanners with Steiner Points,*" *ESA*'20.

Detailed Results

Spread $\Delta = \frac{\max_{p,q \in P} ||p, q||}{\min_{p \neq q \in P} ||p, q||}$

Theorem 1 [LS20]

Given $P \in \mathbb{R}^2$, one can construct a Steiner $(1+\varepsilon)$-spanner G for P with lightness:

$$\text{Lightness}(G) = \frac{w(G)}{w(\text{MST})} = O(\frac{\log \Delta}{\epsilon})$$

Theorem 2 [LS20]

Given $P \in \mathbb{R}^d$, one can construct a Steiner $(1+\varepsilon)$-spanner G for P with lightness:

$$\text{Lightness}(G) = \frac{w(G)}{w(\text{MST})} = \tilde{O}(\epsilon^{(-d-1)/2} + (\log \Delta)\epsilon^{-2})$$

This talk

$$\text{Spread } \Delta = \frac{\max_{p,q \in P} ||p, q||}{\min_{p \neq q \in P} ||p, q||}$$

Theorem 1 [LS20]

Given $P \in \mathbb{R}^2$, one can construct a Steiner $(1+\varepsilon)$-spanner G for P with lightness:

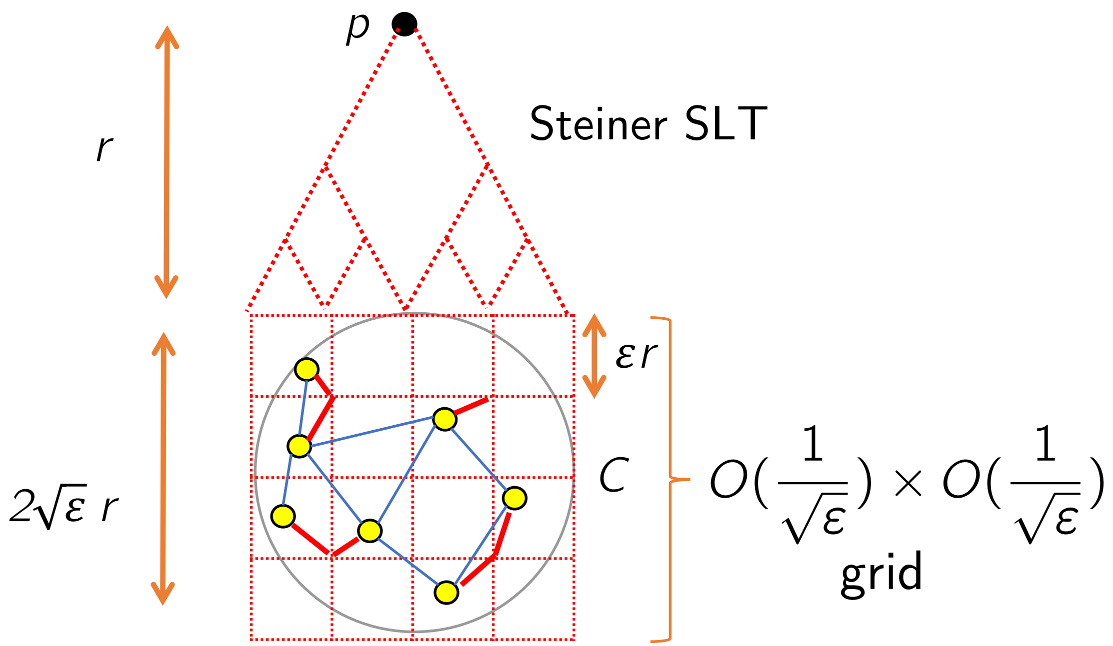
$$\text{Lightness}(G) = \frac{w(G)}{w(\text{MST})} = O(\frac{\log \Delta}{\epsilon})$$

The rest of the talk: proving Theorem 1

Detour: Single Source Spanners (SSP)

Source p, $Q \subseteq C_{(\sqrt{\varepsilon}\, r)}$, $(1+\varepsilon)$-spanner S of Q, $d(p, C_{(\sqrt{\varepsilon}\, r)}) = r$. SSP is a geometric graph H:

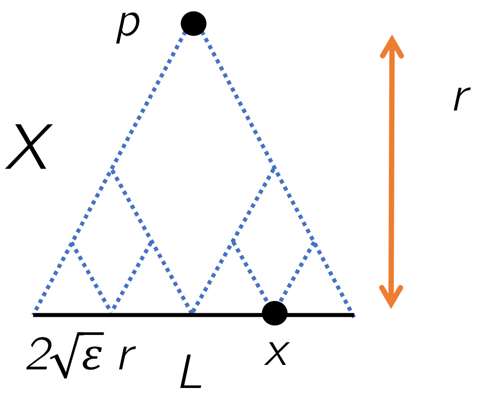
$$d_{H \cup S}(p, x) \leq (1+\varepsilon)||p, x|| \quad \forall x \in Q$$



Steiner SLT: SLT = Shallow Light Tree

point p, line L, $d(p,L) = r$, $length(L) = 2\sqrt{\varepsilon}r$.
Steiner SLT is a geometric graph X:

$$d_{X \cup L}(p, x) \leq (1+\varepsilon)||p, x|| \quad \forall x \in L$$



Solomon [SocG14]: $w(X) = O(r)$

Detour: Single Source Spanners

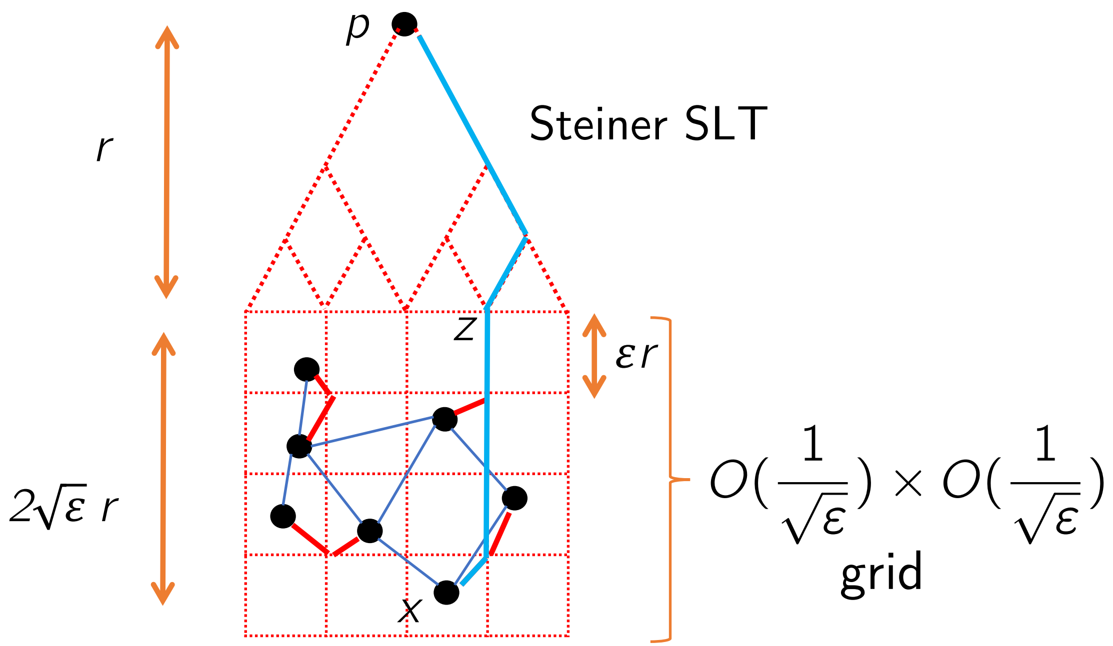
Source p, $Q \subseteq C_{(\sqrt{\varepsilon}\, r)}$, $(1+\varepsilon)$-spanner S of Q,
$d(p, C_{(\sqrt{\varepsilon}\, r)}) = r$. SSP is a geometric graph H:

$$d_{H \cup S}(p, x) \leq (1+\varepsilon)||p, x|| \quad \forall x \in Q$$

Claim: $w(H) = O(r)$

Proof:

$$w(H) = w(\text{SLT}) + w(\text{grid})$$
$$+ w(\text{connections to grid})$$
$$= O(r) + O(r) + O(r)$$

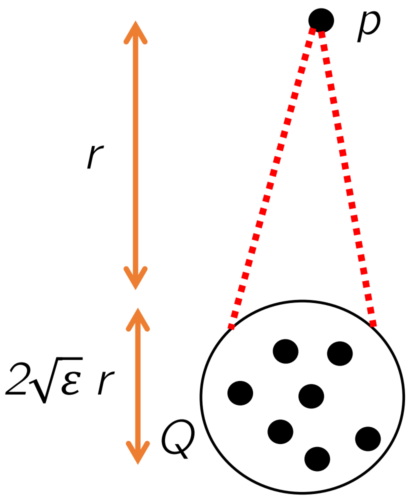


Detour: Single Source Spanners

Lemma: Given a point p, a set of point Q enclosed in a circle C of radius $\sqrt{\varepsilon}r$ where $d(p,C) = r$, and a $(1+\varepsilon)$-spanner S of Q, one can construct a graph H such that:

$$(1) \quad d_{H \cup S}(p,x) \leq (1+\varepsilon)||p,x|| \quad \forall x \in Q$$

$$(2) \quad w(H) = O(r)$$



Steiner Spanner Construction: Overview

Goal: Find a Steiner $(1+\varepsilon)$-spanner G for P with $w(G) = O(\log\Delta/\varepsilon)w(\text{MST})$

- Spread Δ: assume min. distance $= 1$ and max. distance $= \Delta$
- Partition. $\mathcal{P} = \{(p, q)\}_{p \neq q \in P}$ into $log\Delta$ subsets:

$$\mathcal{P}_i = \{(p, q) : ||p, q|| \in (2^{i-1}, 2^i]\}$$

- Find Steiner $(1+\varepsilon)$-spanner H_i for $\mathcal{P}_i$ separately

$$\begin{aligned}
&(1) \quad d_{H_i}(p, q) \leq (1+\epsilon)||p, q|| \quad \forall (p, q) \in \mathcal{P}_i \\
&(2) \quad w(H_i) = O(1/\varepsilon)w(\text{MST})
\end{aligned}$$

Claim: $G = \bigcup_{i=1}^{\log \Delta} H_i$ is the desired spanner.

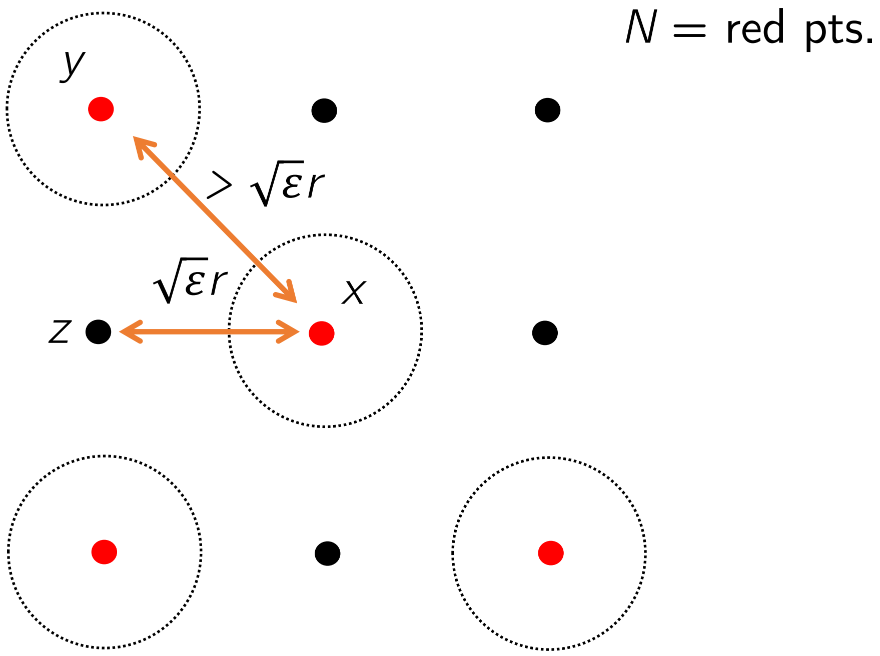
Goal: Find Steiner $(1+\varepsilon)$-spanner H_i for $\mathcal{P}_i = \{(p,q) : ||p,q|| \in (2^{i-1}, 2^i]\}$

$$(1) \quad d_{H_i}(p,q) \le (1+\epsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i$$
$$(2) \quad w(H_i) = O(1/\varepsilon) w(\mathrm{MST})$$

- Let $r = 2^i$, and N be a $(\sqrt{\varepsilon} r)$-net of P:

$$(1) \quad \forall x \ne y \in N, ||x,y|| > \sqrt{\varepsilon} r$$
$$(2) \quad \forall z \in P, \exists x \in N : ||x,z|| \le \sqrt{\varepsilon} r$$

Claim: $|N| = O(\frac{w(\mathrm{MST})}{\sqrt{\varepsilon} r})$



Proof: $\{C(x, \sqrt{\varepsilon} r/2)\}_{x \in N}$ are pairwise disjoint.

$$|N| \frac{\sqrt{\varepsilon} r}{2} \le w(\text{part of MST inside all circles}) \le w(\mathrm{MST})$$

Goal: Find Steiner *(1+ ε)*-spanner H_i for $\mathcal{P}_i = \{(p, q) : ||p, q|| \in (2^{i-1}, 2^i]\}$

$$(1) \quad d_{H_i}(p, q) \leq (1 + \epsilon)||p, q|| \quad \forall (p, q) \in \mathcal{P}_i$$
$$(2) \quad w(H_i) = O(1/\varepsilon) w(\mathrm{MST})$$

- Let $r = 2^i$, and N be a $(\sqrt{\varepsilon} r)$-net of P:

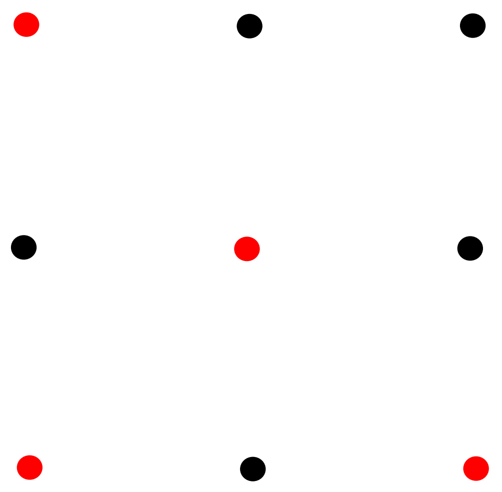
$$(1) \quad \forall x \neq y \in N, ||x, y|| > \sqrt{\varepsilon} r$$
$$(2) \quad \forall z \in P, \exists x \in N : ||x, z|| \leq \sqrt{\varepsilon} r$$

Claim: $|N| = O(\frac{w(\mathrm{MST})}{\sqrt{\varepsilon} r})$

- If $w(H_i) = O(\frac{|N| r}{\sqrt{\epsilon}})$ then:

$$w(H_i) = O(|N| \frac{r}{\sqrt{\varepsilon}}) = O(\frac{w(\mathrm{MST})}{\sqrt{\varepsilon} r} \cdot \frac{r}{\sqrt{\varepsilon}}) = O(\frac{w(\mathrm{MST})}{\varepsilon})$$

$N =$ red pts.



Goal: Find Steiner *(1+ ε)*-spanner H_i for $\mathcal{P}_i = \{(p, q) : ||p, q|| \in (r/2, r]\}$

$$(1) \quad d_{H_i}(p, q) \le (1+\varepsilon)||p, q|| \quad \forall (p, q) \in \mathcal{P}_i$$

$$(2) \quad w(H_i) = O(\frac{|N|r}{\sqrt{\varepsilon}}) \quad N \text{ is } r\sqrt{\varepsilon}\text{-net of } P$$

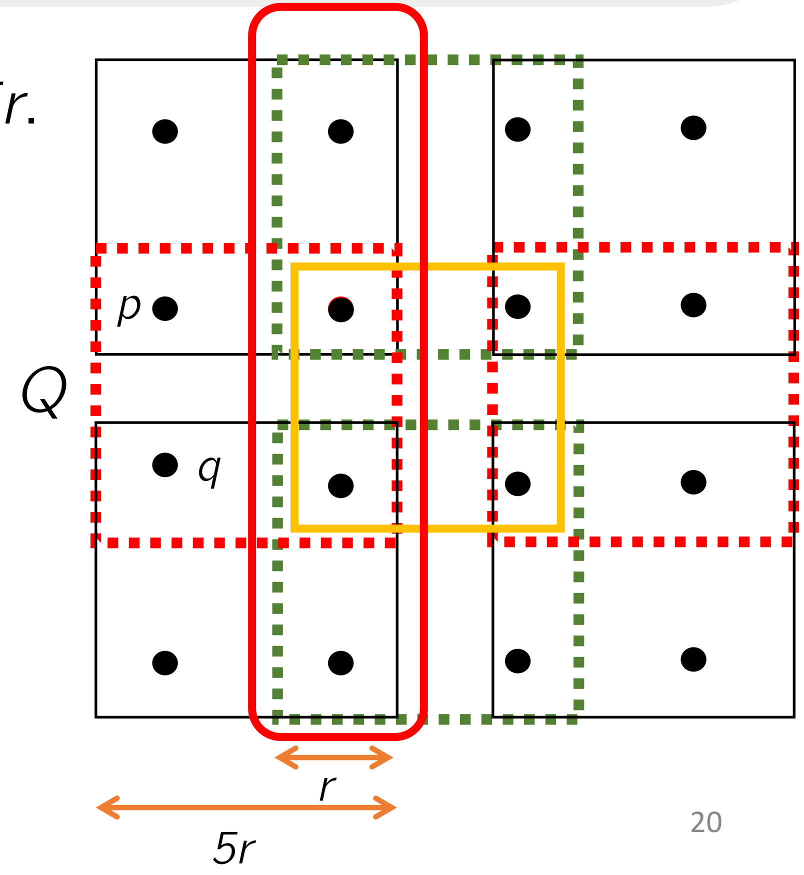
- "Cover" the plane by overlapping □ of size length *5r*.

$(a) \quad \forall x \in N : x \in$ at most 4 □s

$(b) \quad \forall (p, q) : ||p, q|| \le r, \exists$ □ Q s.t $p, q \in Q$

- Focus on 1 square *Q*: preserve *(p,q)* s.t

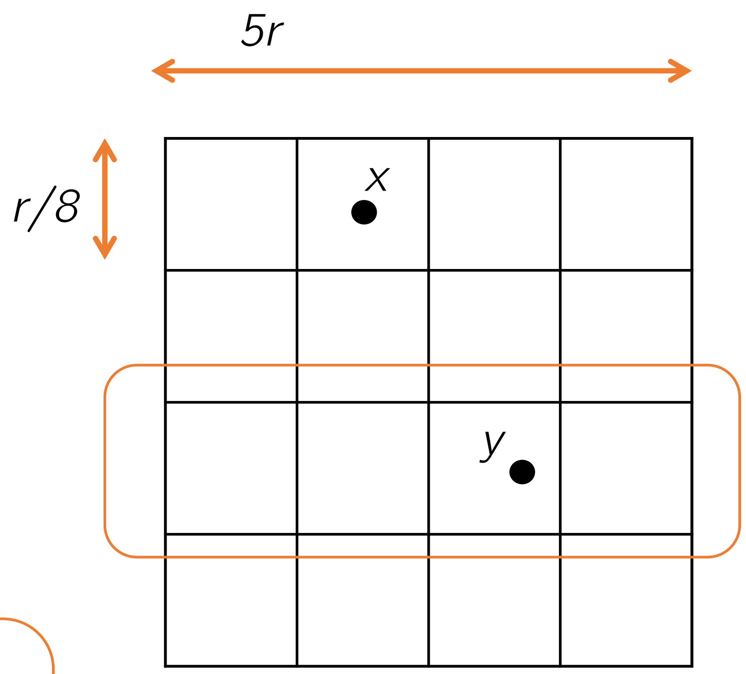
$$(p, q) \in \mathcal{P}_i \cap Q$$



Spanner in a Square

- Place grid with cell size *(r/8)x(r/8)*
- Horizontal band: a row of the grid.
  - O(1) such bands
- Vertical band: a column of the grid

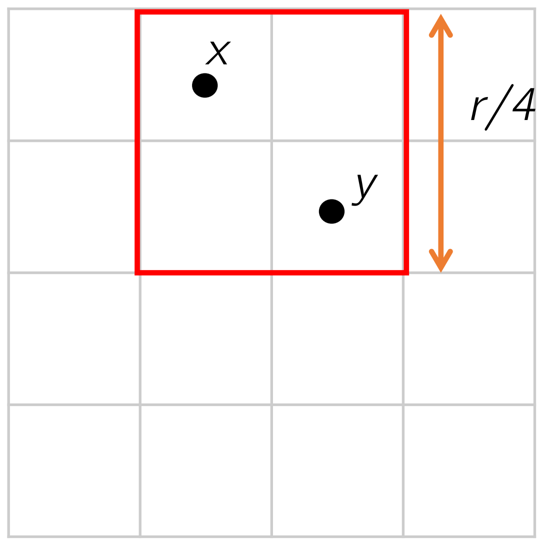
Claim: *(x,y)* ∈ *NxN* such that ||x,y|| ∈ *(r/2,r]*. Then:
- x,y in two **non-adjacent horizontal** bands
- or x,y in two **non-adjacent vertical** bands



Proof:

Otherwise, x,y both in ▢ of size length *r/4.*

$$||x, y|| \leq \sqrt{2} \cdot \frac{r}{4} < \frac{r}{2}$$



Spanner in a Square (cont.)

SquareSpanner(Q)

Divide Q into a grid with cell size $(r/8)x(r/8)$

$H_Q \leftarrow \emptyset$ // square spanner

Foreach pair (X,Y) of non-adj. horizontal (vertical) bands

    $H_{X,Y} \leftarrow \emptyset$ // band spanner

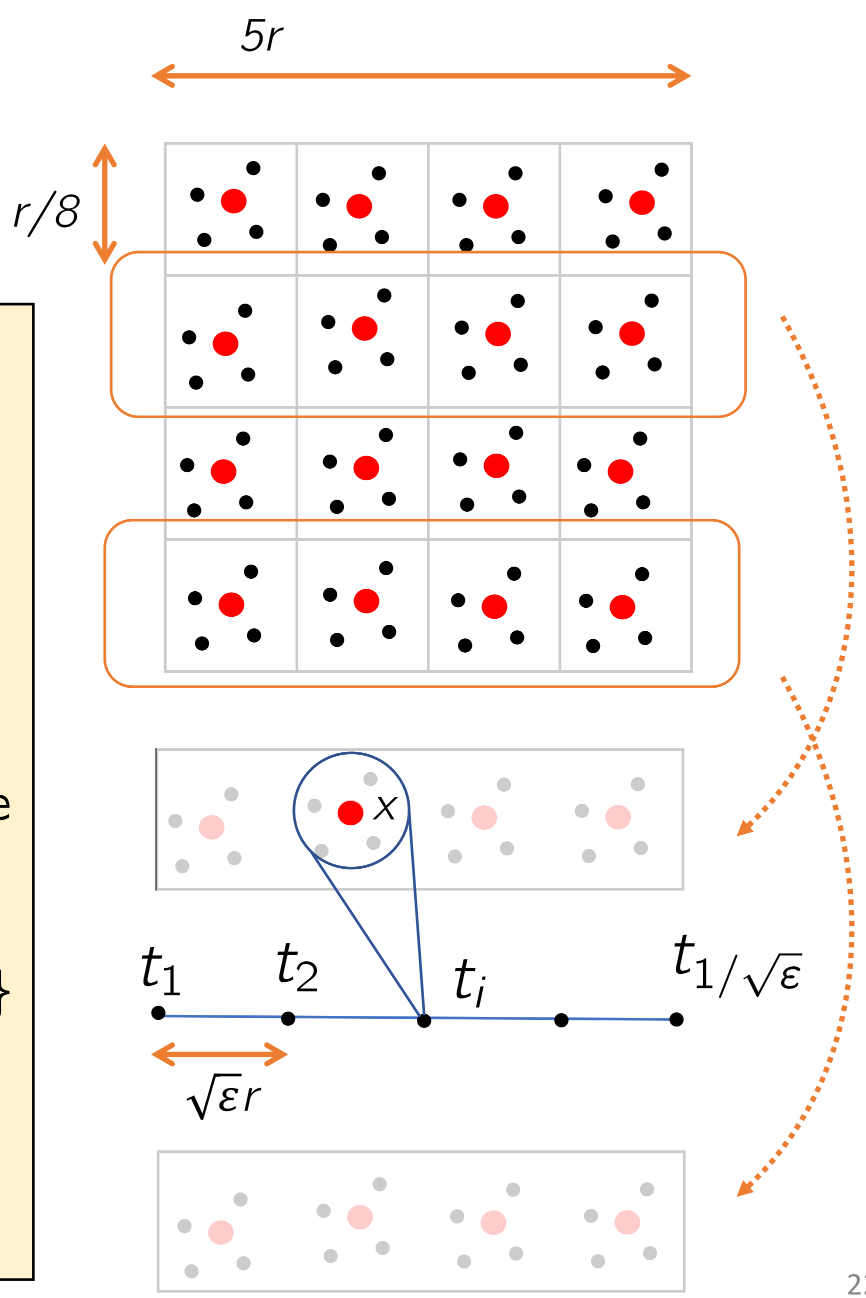
    Place $O(1/\sqrt{\varepsilon})$ Steiner points $\{t_1, t_2 \ldots\}$ on the mid-line

    Foreach point t_i and each point x in $N \cap (X \cup Y)$:

        $H_{X,Y} \leftarrow H_{X,Y} \cup$ {a SS spanner from t_i to $C(x, \sqrt{\varepsilon} r)$}

    $H_Q \leftarrow H_Q \cup H_{X,Y}$

return H_Q



Spanner in a Square (cont.)

SquareSpanner(Q)

1. Divide Q into a grid with cell size $(r/8)x(r/8)$
2. $H_Q \leftarrow \emptyset$ // square spanner
3. **Foreach** pair (X,Y) of non-adj. horizontal (vertical) bands → $O(1)$ pairs
4. $H_{X,Y} \leftarrow \emptyset$ // band spanner
5. Place $O(1/\sqrt{\varepsilon})$ Steiner points $\{t_1, t_2 \ldots\}$ on the mid-line
6. **Foreach** point t_i and each point x in $N \cap (X \cup Y)$:
7. $H_{X,Y} \leftarrow H_{X,Y} \cup$ {a SS spanner from t_i to $C(x, \sqrt{\varepsilon} r)$} — $O(r)$ weight
8. $H_Q \leftarrow H_Q \cup H_{X,Y}$
9. return H_Q

Claim: $w(H_Q) = O(\frac{r}{\sqrt{\varepsilon}}|N \cap Q|)$

Proof:

$O(1)$ pairs

$$w(H_{X,Y}) = O(\frac{1}{\sqrt{\varepsilon}})r|N \cap (X \cup Y)|$$

$$w(H_Q) = O(\frac{r}{\sqrt{\varepsilon}})(\sum_{X,Y} |N \cap (X \cup Y)|)$$

$$= O(\frac{r}{\sqrt{\varepsilon}})|N \cap Q|$$

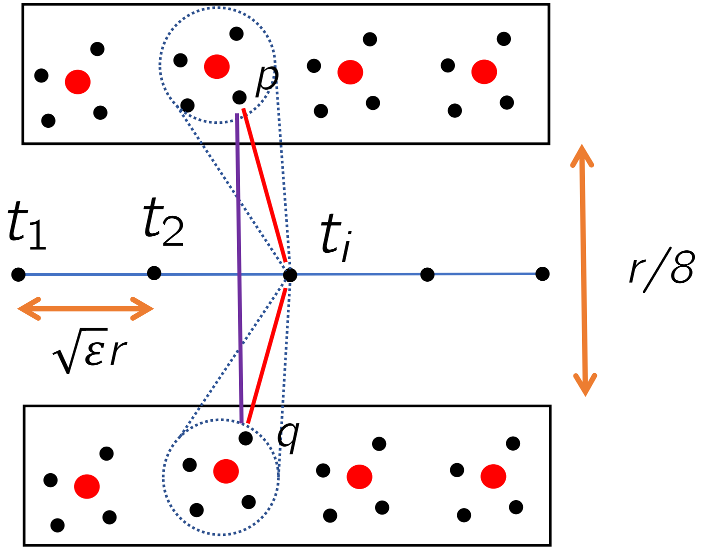
Spanner in a Square (cont.)

SquareSpanner(Q)

1. Divide Q into a grid with cell size $(r/8)x(r/8)$
2. $H_Q \leftarrow \emptyset$ // square spanner
3. **Foreach** pair (X,Y) of non-adj. horizontal (vertical) bands
4. $H_{X,Y} \leftarrow \emptyset$ // band spanner
5. Place O($1/\sqrt{\varepsilon}$) Steiner points $\{t_1, t_2 \ldots\}$ on the mid-line
6. **Foreach** point t_i and each point x in $N \cap (X \cup Y)$:
7. $H_{X,Y} \leftarrow H_{X,Y} \cup$ {a SS spanner from t_i to $C(x, \sqrt{\varepsilon} r)$}
8. $H_Q \leftarrow H_Q \cup H_{X,Y}$
9. return H_Q

Claim: $w(H_Q) = O(\frac{r}{\sqrt{\varepsilon}}|N \cap Q|)$

Claim: any $(p, q) \in \mathcal{P}_i \cap Q$
then: $d_{H_Q}(p, q) \leq (1 + \varepsilon)||p, q||$



Goal: Find Steiner $(1+\varepsilon)$-spanner H_i for $\mathcal{P}_i = \{(p,q) : ||p,q|| \in (r/2, r]\}$

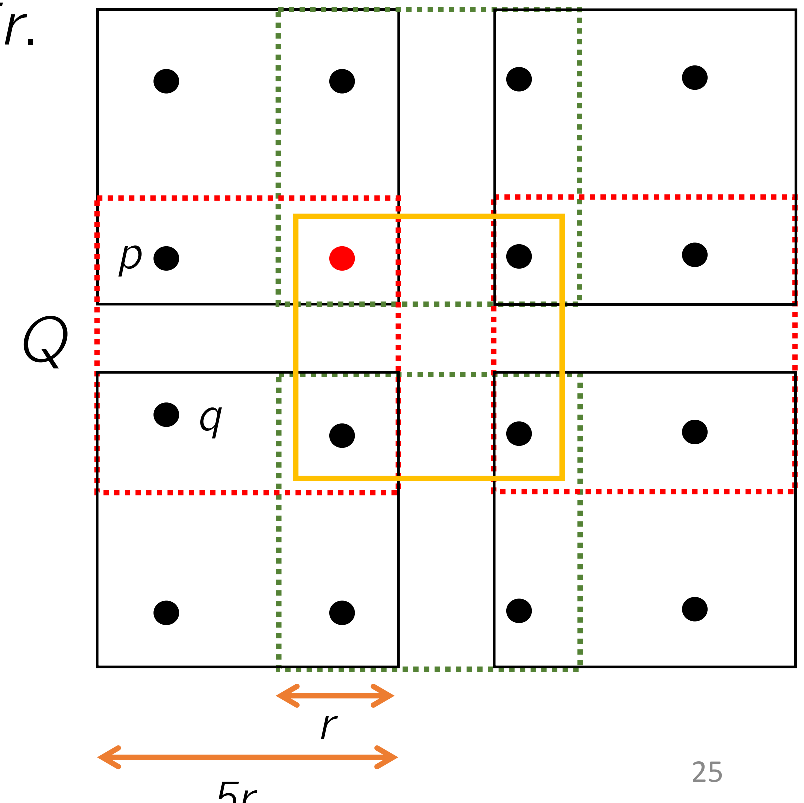
$$(1) \quad d_{H_i}(p,q) \le (1+\varepsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i$$

$$(2) \quad w(H_i) = O(\frac{|N|r}{\sqrt{\varepsilon}}) \quad N \text{ is } r\sqrt{\varepsilon}\text{-net of } P$$

- "Cover" the plane by overlapping ▢ of size length $5r$.
  - $(a) \quad \forall x \in N : x \in$ at most 4 □s
  - $(b) \quad \forall (p,q) : ||p,q|| \le r, \exists$ □ Q s.t $p, q \in Q$
- For each square Q: $w(H_Q) = O(\frac{r}{\sqrt{\varepsilon}}|N \cap Q|)$

  $$d_{H_Q}(p,q) \le (1+\varepsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i \cap Q$$
- Let

$$H_i = \bigcup_Q H_Q$$



Goal: Find Steiner *(1+ ε)*-spanner H_i for $\mathcal{P}_i = \{(p,q) : ||p,q|| \in (r/2, r]\}$

(1) $d_{H_i}(p,q) \le (1+\varepsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i$ ✓

(2) $w(H_i) = O(\frac{|N|r}{\sqrt{\varepsilon}}) \quad N$ is $r\sqrt{\varepsilon}$-net of P ✓

- "Cover" the plane by overlapping ▢ of size length *5r*.

  (a) $\forall x \in N : x \in$ at most 4 ☐s

  (b) $\forall (p,q) : ||p,q|| \le r, \exists$ ☐ Q s.t $p,q \in Q$

- For each square Q: $w(H_Q) = O(\frac{r}{\sqrt{\varepsilon}}|N \cap Q|)$

  $d_{H_Q}(p,q) \le (1+\varepsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i \cap Q$

- Let

$$H_i = \bigcup_Q H_Q$$

$$w(H_i) = O(\frac{r}{\sqrt{\varepsilon}}) \sum_Q |N \cap Q|$$

$$= O(\frac{r|N|}{\sqrt{\varepsilon}})$$

$$d_{H_i}(p,q) \le (1+\varepsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i$$

Steiner Spanner Construction: Overview

Goal: Find a Steiner $(1+\varepsilon)$-spanner G for P with $w(G) = O(\log\Delta/\varepsilon)w(\text{MST})$

- Spread Δ: assume min. distance $= 1$ and max. distance $= \Delta$
- Partition. $\mathcal{P} = \{(p,q)\}_{p \neq q \in P}$ into $log\Delta$ subsets:

$$\mathcal{P}_i = \{(p,q) : ||p,q|| \in (2^{i-1}, 2^i]\}$$

- Find Steiner $(1+\varepsilon)$-spanner H_i for $\mathcal{P}_i$

$$\begin{array}{ll} (1) & d_{H_i}(p,q) \leq (1+\epsilon)||p,q|| \quad \forall (p,q) \in \mathcal{P}_i \\ (2) & w(H_i) = O(1/\varepsilon)w(\text{MST}) \end{array}$$

✓

Claim: $G = \bigcup_{i=1}^{\log\Delta} H_i$ is the desired spanner.

Our results (recap)

Spread $\Delta = \frac{\max_{p,q \in P} ||p,q||}{\min_{p \neq q \in P} ||p,q||}$

Theorem 1 [LS20]

Given $P \in \mathbb{R}^2$, one can construct a Steiner $(1+\varepsilon)$-spanner G for P with lightness:

$$\text{Lightness}(G) = \frac{w(G)}{w(\text{MST})} = O\left(\frac{\log \Delta}{\epsilon}\right)$$

Open Problems

$t = 1 + \varepsilon$

$\tilde{\Omega}$, $\tilde{O}$ hide $polylog(1/\varepsilon)$.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(1/\varepsilon)$ [Clarkson87;Keil88]	$\Omega(1/\varepsilon)$ [LS,FOCS19]	$\tilde{O}(1/\sqrt{\varepsilon})$ [LS,FOCS19]	$\tilde{\Omega}(1/\sqrt{\varepsilon})$ [LS,FOCS19]
Lightness	$\tilde{O}(1/\varepsilon^2)$ [LS,FOCS19]	$\Omega(1/\varepsilon^2)$ [LS,FOCS19]	$\tilde{O}(1/\varepsilon)$ [LS,ESA20]	$\tilde{\Omega}(1/\varepsilon)$ [LS,FOCS19]

when spread Δ is $poly(1/\varepsilon)$

Problem 1: Find a Steiner $(1+\varepsilon)$-spanner G for P with $w(G) = O(1/\varepsilon)w(\text{MST})$ for any spread in Euclidean plane.

Open Problems

$t = 1 + \varepsilon$

$\tilde{\Omega}$, $\tilde{O}$ hide $polylog(1/\varepsilon)$.

	Non-Steiner Spanners		Steiner Spanners	
Sparsity	$O(\varepsilon^{1-d})$ [RS91]	$\Omega(\varepsilon^{1-d})$ [LS,FOCS19]	$\tilde{O}(\varepsilon^{(1-d)/2})$ [LS,FOCS19]	? Lower bound
Lightness	$\tilde{O}(\varepsilon^{-d})$ [LS,FOCS19]	$\Omega(\varepsilon^{-d})$ [LS,FOCS19]	$\tilde{O}(\varepsilon^{(-d-1)/2})$ [LS20]	? Lower bound

for any spread

[LS20] H. Le and S. Solomon, "*A Unified and Fine-Grained Approach for Light Spanners*", *Preprint*.

Thank you

Q&A